Self-assembling tensor networks and holography in disordered spin chains

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Disordered Heisenberg chain

\[ H = \sum_{i=1}^{L=1} J_i \vec{s}_i \cdot \vec{s}_{i+1} \]

AFM: \( 0 < J_i \leq J_{\text{max}} \)

• Paradigmatic model for interplay of disorder and many-body (spin-spin) interactions

• Much is known already, both theoretically and numerically: \textit{ideal test case!}
Disordered Heisenberg chain

- Theoretical results: Ma, Dasgupta, Hu
  - strong disorder real space RG (SDRG): successive elimination of neighboring spins with maximal coupling $J_{\text{max}}$
  - Universal power-law dependences for specific heat $C \propto T^{\gamma_c}$ and susceptibility $\chi \propto T^{\gamma_s - 1}$

$J_{i-1} \quad J_i \quad J_{i+1} \quad s_{i-1} \quad s_i \quad s_{i+1} \quad s_{i+2}$

renormalize

$\tilde{J}$

$\tilde{J} = J_{i-1}J_{i+1} / 2J_i$

$S_{i-1} \quad S_i \quad S_{i+1} \quad S_{i+2}$

Disordered Heisenberg chain

- Theoretical results: Fisher; Refael+Moore

  - MDH ground state is “random singlet phase”
  - Spin-spin correlation power-law
  - Mean $\langle \vec{s}_i \cdot \vec{s}_j \rangle$, not typical

- Entanglement

$$S_{A,B} \sim \frac{\log 2}{3} \log_2 L_B \approx 0.231 \log L_B$$

[D. S. Fisher, PRB 50, 3799 (1994); PRB 51, 6411 (1995); G. Refael and J. E. Moore, PRL 93, 260602 (2004)]
Disordered Heisenberg chain

- **Extended strategies:**
  - use (largest) energy gaps $\Delta$
  - works for spin beyond $s=1/2$
  - universal for most $P(J)$

\[
\Delta_i = J_i \left( |s_i - s_{i+1}| + 1 \right)
\]

\[
C \propto T^{-0.88} |\ln T|
\]

\[
\chi \propto T^{-1}
\]

Disordered Heisenberg chain

- Numerical strategies: Hikihara et al.
  - higher multiplet excitations
  - numerical verification of

\[
\langle \vec{S}_i \cdot \vec{S}_j \rangle \propto \frac{(-1)^{i-j}}{|i-j|^2}
\]

- renormalize
- keep more than ground state

The gold standard: DMRG

Exponential growth of Hilbert space avoided by truncating using local information while keeping non-trivial quantumness of states

- Infinite and finite-system algorithms

MPS, MPO formulation of DMRG

- DMRG builds up a matrix-product state

\[ |\Psi\rangle = \sum_{\sigma_1, \ldots, \sigma_L} C_{\sigma_1, \ldots, \sigma_L} |\sigma_1, \ldots, \sigma_L\rangle \]

\[ \approx \sum_{\sigma_1, \ldots, \sigma_L} \sum_{a_1, \ldots, a_L} M_{a_1}^{\sigma_1} M_{a_1, a_2}^{\sigma_2} \cdots M_{a_{L-2}, a_{L-1}}^{\sigma_{L-1}} M_{a_{L-1}}^{\sigma_L} |\sigma_1, \ldots, \sigma_L\rangle \]

- Physical indices (spin) \( \sigma_1, \ldots, \sigma_L \)

- Tensor \( C_{\sigma_1, \ldots, \sigma_L} \) can be decomposed into tensor network.

MPS, **MPO** formulation of DMRG

• **Operators** are written similarly for consistency

\[
O = \sum_{\sigma_1,\ldots,\sigma_L} \sum_{\tau_1,\ldots,\tau_L} D_{\sigma_1,\tau_1,\ldots,\sigma_L,\tau_L} \langle \sigma_1,\ldots,\sigma_L \rangle \langle \tau_1,\ldots,\tau_L \rangle
\]

\[
= \sum_{\sigma_1,\ldots,\sigma_L} \sum_{\tau_1,\ldots,\tau_L} \sum_{b_1,\ldots,b_L} W_{b_1}^{\sigma_1,\tau_1} W_{b_1,b_2}^{\sigma_2,\tau_2} \cdots W_{b_{L-2},b_{L-1}}^{\sigma_{L-1},\tau_{L-1}} W_{b_{L-1}}^{\sigma_L,\tau_L} \times
\]

\[
| \sigma_1,\ldots,\sigma_L \rangle \langle \tau_1,\ldots,\tau_L |
\]

MPOs in practice - an example:

\[
W_{b_2,b_3} W_{b_3,b_4} = \begin{pmatrix}
1 & \frac{J_3}{2} s_3^+ & \frac{J_3}{2} s_3^- & J_3 s_3^z & 0 \\
0 & 0 & 0 & 0 & s_3^- \\
0 & 0 & 0 & 0 & s_3^+ \\
0 & 0 & 0 & 0 & s_3^z \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & \frac{J_4}{2} s_4^+ & \frac{J_4}{2} s_4^- & J_4 s_4^z & 0 \\
0 & 0 & 0 & 0 & s_4^- \\
0 & 0 & 0 & 0 & s_4^+ \\
0 & 0 & 0 & 0 & s_4^z \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & \frac{J_3}{2} s_3^+ & \frac{J_3}{2} s_3^- & J_3 s_3^z & 0 \\
0 & 0 & 0 & 0 & s_3^- \\
0 & 0 & 0 & 0 & s_3^+ \\
0 & 0 & 0 & 0 & s_3^z \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & \frac{J_4}{2} s_4^+ & \frac{J_4}{2} s_4^- & J_4 s_4^z & 0 \\
0 & 0 & 0 & 0 & s_4^- \\
0 & 0 & 0 & 0 & s_4^+ \\
0 & 0 & 0 & 0 & s_4^z \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & \frac{J_3}{2} s_3^+ & \frac{J_4}{2} s_4^- & \frac{J_3}{2} s_3^+ s_4^- + \frac{J_3}{2} s_3^- s_4^+ + J_3 s_3^z s_4^z & 0 \\
0 & 0 & 0 & 0 & s_4^- \\
0 & 0 & 0 & 0 & s_4^+ \\
0 & 0 & 0 & 0 & s_4^z \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

MPS, MPO formulation of DMRG

- Moving to cartoons

\[ |\Psi\rangle = \sum_{\sigma_1, \ldots, \sigma_L} \sum_{a_1, \ldots, a_L} M_{\sigma_1}^{a_1} M_{\sigma_2}^{a_2} \cdots M_{\sigma_{L-1}}^{a_{L-2}, a_{L-1}} M_{\sigma_L}^{a_L} |\sigma_1, \ldots, \sigma_L\rangle \]

\[ O = \sum_{\sigma_1, \ldots, \sigma_L} \sum_{\tau_1, \ldots, \tau_L} \sum_{b_1, \ldots, b_L} W_{b_1}^{\sigma_1, \tau_1} W_{b_1, b_2}^{\sigma_2, \tau_2} \cdots W_{b_{L-2}, b_{L-1}}^{\sigma_{L-1}, \tau_{L-1}} W_{b_{L-1}}^{\sigma_L, \tau_L} |\sigma_1, \ldots, \sigma_L\rangle \langle \tau_1, \ldots, \tau_L| \]

MPS, MPO formulation of DMRG

- Moving to cartoons

\[ |\Psi\rangle = \sum_{\sigma_1, \ldots, \sigma_L, a_1, \ldots, a_L} M_{a_1}^{\sigma_1} M_{a_2}^{\sigma_2} \cdots M_{a_{L-1}}^{\sigma_{L-1}} M_{a_L}^{\sigma_L} |\sigma_1, \ldots, \sigma_L\rangle \]

\[ \langle \Psi | H | \Psi \rangle = \sum_{\sigma_1, \ldots, \sigma_L, \sigma_1', \ldots, \sigma_L'} \sum_{a_1, \ldots, a_L, a_1', \ldots, a_L'} \sum_{b_1, \ldots, b_L} M_{a_1}^{\sigma_1} W_{1,b_1}^{\sigma_1,\sigma_1'} M_{a_1,a_2}^{\sigma_1} M_{a_2}^{\sigma_2} W_{b_1,b_2}^{\sigma_2,\sigma_2'} M_{a_2,a_3}^{\sigma_2} \cdots \]

\[ \cdots M_{a_{L-1},a_L}^{\sigma_{L-1}} W_{b_{L-1},b_L}^{\sigma_{L-1},\sigma_{L-1}'} M_{a_{L-1}}^{\sigma_{L-1}} M_{a_L}^{\sigma_L} W_{b_{L-1},b_L}^{\sigma_L,\sigma_L'} M_{a_L}^{a_L'} \]

MPS, MPO formulation of DMRG

- Reformulation of 2-site DMRG
- Density matrix Schur-decomposed and truncated

\[ A = UDV^\dagger \]

\[ \chi = \text{bond dimension} \]
• Contract MPO tensors for sites with largest gap

• Keep lowest eigenvalues only and contract

SDRG MPO process

• Contract MPO tensors for sites with largest gap

\[ W[i_m, i_{m+1}] = \sum_{b_{im}} W_{b_{im-1}, b_{im}} W_{b_{im}, b_{im+1}}^{\sigma_{im}, \sigma_{im}'} \]

• Keep lowest \( \chi \) eigenvalues only and contract

\[ \Delta \chi = V^\dagger \left( H_i^B \otimes 1 + J_i^m \vec{s}_i^R \cdot \vec{s}_i^L + 1 \otimes H_i^{B_{m+1}} \right) V \chi \]

\[ W_{\tilde{\sigma}_{im-1}, \tilde{\sigma}_{im}} = \sum_{\tau, \tau'} [V^\dagger \chi]_{\tau} W_{\tau, \tau'} \left[ V \chi \right]_{\tau'}^{\tilde{\sigma}_{im}} \]

SDRG as TTN

Basic RG step:

\[ w = \tilde{\sigma}_m \]

Isometry:

\[ w = \tilde{\sigma}_m \]

Hilbert space

Lattice Dimension
Tree tensor networks pictorially
Why bother?

- \( \text{vMPS works well for clean systems} \)
- \( \text{vMPS ignores disorder in setup and hence update process is highly affected} \)
- \( \text{Inhomogeneous TTN as presented here includes disorder as building block of network itself, ADAPTIVE} \)
Ground state convergence

DMRG = vMPS
SDRG with TTN = tSDRG

- tSDRG converges
- Energies a bit better in vMPS
- Larger disorder improves tSDRG

\[ J_i \in \left[ 1 - \frac{\Delta J}{2}, 1 + \frac{\Delta J}{2} \right] \]
Spin-spin correlation

- Fisher $1/r^2$ recovered
- Large distance behaviour dominated by boundaries

$$\langle \vec{s}_i \cdot \vec{s}_j \rangle \propto \frac{1}{|i-j|^2}$$
The idea of “holography”

- **Maldacena/Witten**: physical world lives on surface of larger/bulk space
- **Swingle**: tensor networks are like larger space
- **Evenbly/Vidal**: might help in computing correlations
  - Causal cone

\[
\langle \mathcal{O}(x_i)\mathcal{O}(x_j) \rangle \sim \exp[-\Delta \text{length}_\gamma]
\]

\[
S_B = \frac{\text{area}_\gamma}{4G_N^{d+2}}
\]

[minimal surface \(\gamma\)]

[holographic dimension]

Fitting a path length

- Proposal:

\[
\left\langle \vec{S}_{x_1} \cdot \vec{S}_{x_2} \right\rangle \propto e^{-\alpha D(x_1, x_2)} \propto e^{-\alpha \beta \log |x_1 - x_2|} \propto |x_1 - x_2|^{-\alpha}
\]
Spin-spin correlation as a holographic path length

\[ a = \alpha \beta = (0.62 \pm 0.01)(2.94 \pm 0.02) = 1.84 \pm 0.04 \]

\[ L = 150 \]
Entanglement entropy

- Holography suggests new method of calculation:

\[ S_{A|B} \propto n_A \]

minimum \# bonds that need to be “cut” to separate A from B

Area law, see [Eisert et al, RMP 82, 277 (2010)]

[idea does not work in vMPS]
Entanglement entropy

- vMPS best for small $\Delta J$ disorder
- tSDRG better for $\Delta J > 1$
- Large difference between mean and maximum $S$

$$S_{A|B} = -Tr \rho_A \log_2 \rho_A$$
Log($L$) increase for $S_{A|B}$

- $v$MPS needs to increase $\chi$
- $t$SDRG does not, much easier to compute for large $L$
- Large difference between mean and maximum $S$

$\Delta J = 2$
Block entanglement $S_{A,B}$

- Refael/Moore result implies CFT with effective central charge $\tilde{c} = 1 \cdot \log 2$
- We find

$$S_{A,B} \approx 0.230 \log L_B$$

\[S_{A,B} \sim \frac{\log 2}{3} \log_2 L_B \approx 0.231 \log L_B\]
Entanglement per bond, $S/n_A$

- for $S_{A|B}$ and $S_{A,B}$
- we find constant ratio in bulk

$$\frac{S}{n_A} \approx 0.47 \pm 0.02$$

- $S / n_A = 1/2$
- implies

$$n_A = 2 \log L_B / 3$$
What about other tree tensor networks?

Can we perhaps compute the path lengths explicitly for some tree networks?
Path lengths in $m$-ary trees

- Full and complete $m$-ary trees have been well studied, e.g. for sorting problems.
- But 1D order imposes an apparently not yet studied condition.

Path lengths in $m$-ary trees

- We find

\[
L_{q,n}^{(m)}(r) = \frac{M_{q,n}^{(m)}(r)}{m^n - r}
\]

\[
\Phi(m,q,n) = \text{Hurwitz-Lerch transcendent}
\]

\[
M_{q,n}^{(m)}(r) = m^{n-n_c} 2^q n^q (m^{n_c} - r) + r(m-1)(-2)^q \left[ \Phi(m,-q,-n) - m^{n-n_c} \Phi(m,-q,-n_c) \right]
\]

\[
n_c^{(m)} = \left\lfloor \log_m(r) \right\rfloor + 1
\]

Path lengths in *Catalan* trees

- Full, but not complete
- Catalan numbers 1, 1, 2, 5, 14, ...
- Not a random tree

Path lengths in Catalan trees

We find

\[(n - r + 1)D_n^r = S_n(r)\]

\[A_n(r) = \frac{D_n^r}{C_n}\]

Random binary trees

- not full, nor complete
- $n!$ possibilities
- Computed for 500 samples of size $L=10, 100, 500, 1000$
The networks are different, so the physics will be as well ... (?)
Periodic boundaries (binary)

*Circle Limit III* is a woodcut made in 1959 by Dutch artist *M. C. Escher*
Conclusions

• An inhomogeneous, self-assembled TTN works. First such beast ever!

• We compared $v$MPS with $v^t$SDRG; $v^t$SDRG is possible and should be much better (just as finite-size DMRG is better than infinite-size DMRG)

• Holography holds in 1D disordered quantum spin chains, 1$^\text{st}$ quantitative check!

• Explicit construction of path lengths for a variety of tree graphs: Fun with trees!

Isometries make computation faster
How to do the RG with a TTN