Self-assembling tensor networks and holography in disordered spin chains

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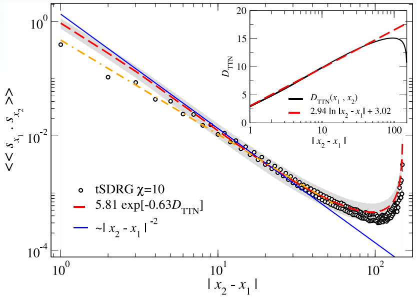
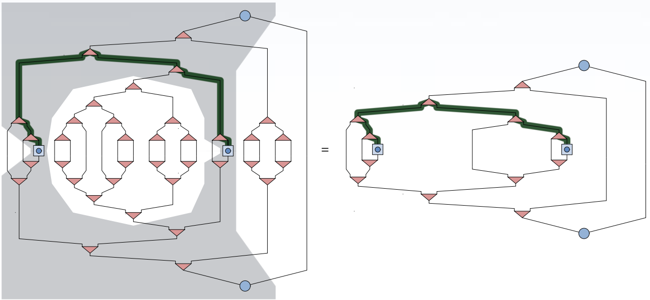
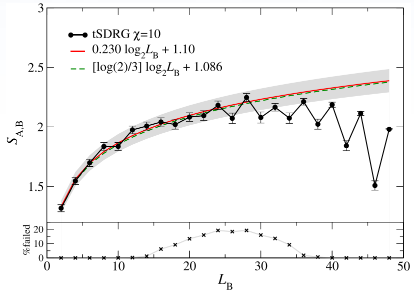
- **A. M. Goldsborough**, RAR, Phys. Rev. B **89**, 214203-11 (2014)
- “Leaf-to-leaf distances and their moments in finite and infinite m-ary tree graphs”, **A.M. Goldsborough**, S.A. Rautu, RAR, arXiv:1406.4079
- “Leaf-to-leaf distances in ordered Catalan tree graphs”, **A.M. Goldsborough, J.M. Fellows**, M. Bates, S.A. Rautu, G. Rowlands and RAR, soon on arXiv







Disordered Heisenberg chain

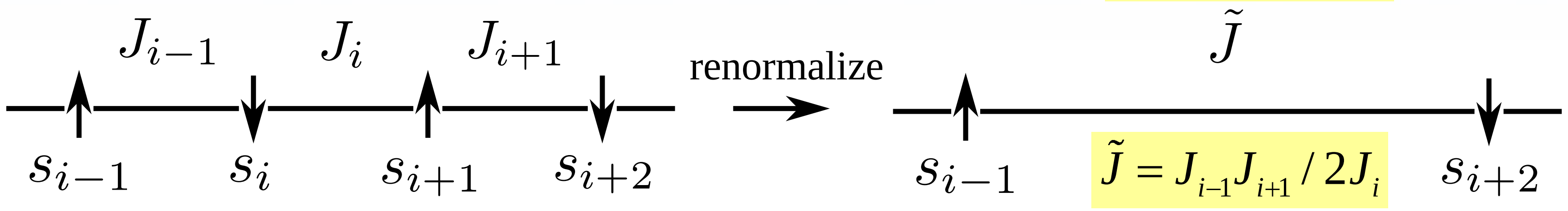
$$H = \sum_{i=1}^{L=1} J_i \vec{s}_i \bullet \vec{s}_{i+1}$$

$$\mathrm{AFM}: 0 < J_i \leq J_{\max}$$

- Paradigmatic model for interplay of disorder and many-body (spin-spin) interactions
- Much is known already, both theoretically and numerically: ideal test case!

Disordered Heisenberg chain

- Theoretical results: Ma, Dasgupta, Hu
  - strong disorder real space RG (SDRG): successive elimination of neighboring spins with maximal coupling $J_{\max}$
  - Universal power-law dependences for specific heat $C \propto T^{\gamma_c}$ and susceptibility $\chi \propto T^{\gamma_s - 1}$



[S. K. Ma, C. Dasgupta, and C. K. Hu, Phys. Rev. Lett. 43, 1434 (1979); C. Dasgupta and S. K. Ma, Phys. Rev. B 22, 1305 (1980)]

Disordered Heisenberg chain

- Theoretical results: Fisher; Refael+Moore
  - MDH ground state is "random singlet phase"
  - Spin-spin correlation power-law $\overline{\left\langle \vec{s}_i \bullet \vec{s}_j \right\rangle} \propto \dfrac{(-1)^{i-j}}{|i-j|^2}$
  - Mean $\overline{\left\langle \vec{s}_i \bullet \vec{s}_j \right\rangle}$, not typical
  - Entanglement $S_{A,B} \sim \dfrac{\log 2}{3} \log_2 L_B \approx 0.231 \log L_B$

[D. S. Fisher, PRB 50, 3799 (1994); PRB 51, 6411 (1995); G. Refael and J. E. Moore, PRL 93, 260602 (2004)]

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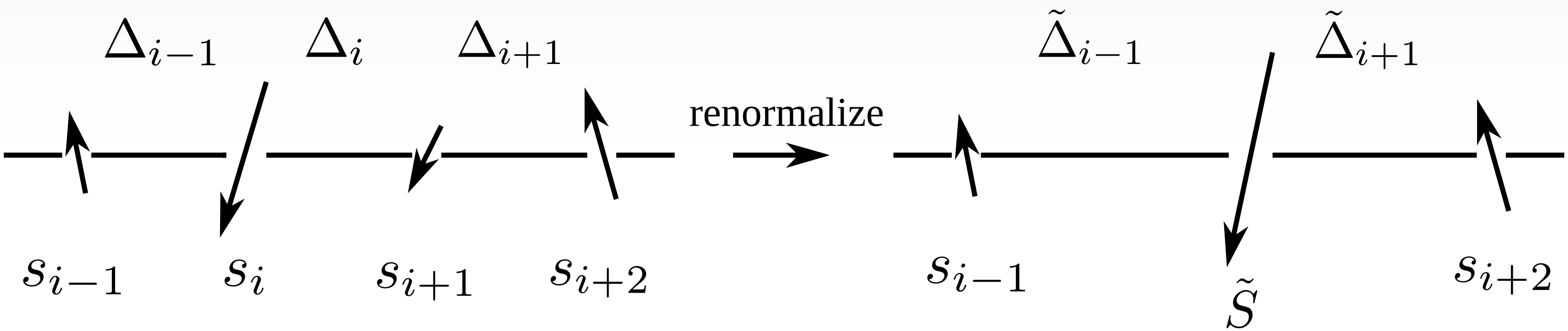
Disordered Heisenberg chain

- Extended strategies:
  - use (largest) energy gaps Δ
  - works for spin beyond s=1/2
  - universal for most P(J)

Westerberg et al.

$$C \propto T^{-0.88} \left|\ln T\right|$$

$$\chi \propto T^{-1}$$



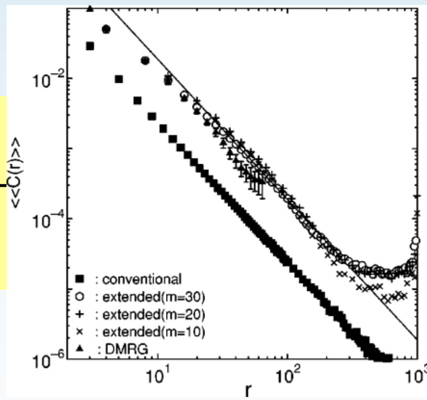
[E. Westerberg, A. Furusaki, M. Sigrist, and P. A. Lee, Phys. Rev. Lett. 75, 4302 (1995); Phys. Rev. B 55, 12578 (1997).]

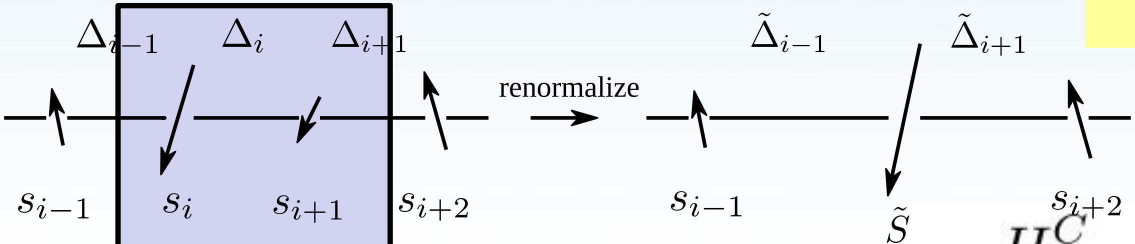
$$\Delta_i = J_i \left(\left| s_i - s_{i+1} \right| + 1 \right)$$

Disordered Heisenberg chain

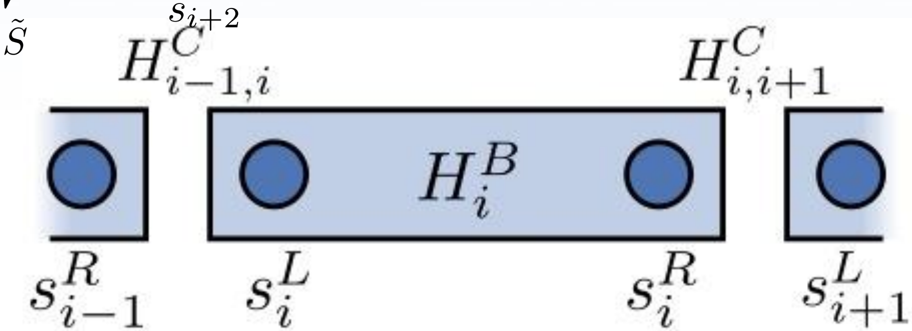
- Numerical strategies: Hikihara et al.
  - higher multiplet excitations
  - numerical verification of

$$\overline{\left\langle \vec{s}_i \bullet \vec{s}_j \right\rangle} \propto \frac{(-1)^{i-j}}{|i-j|^2}$$





keep more than ground state



[T. Hikihara, A. Furusaki, and M. Sigrist, Phys. Rev. B 60, 12116 (1999).]

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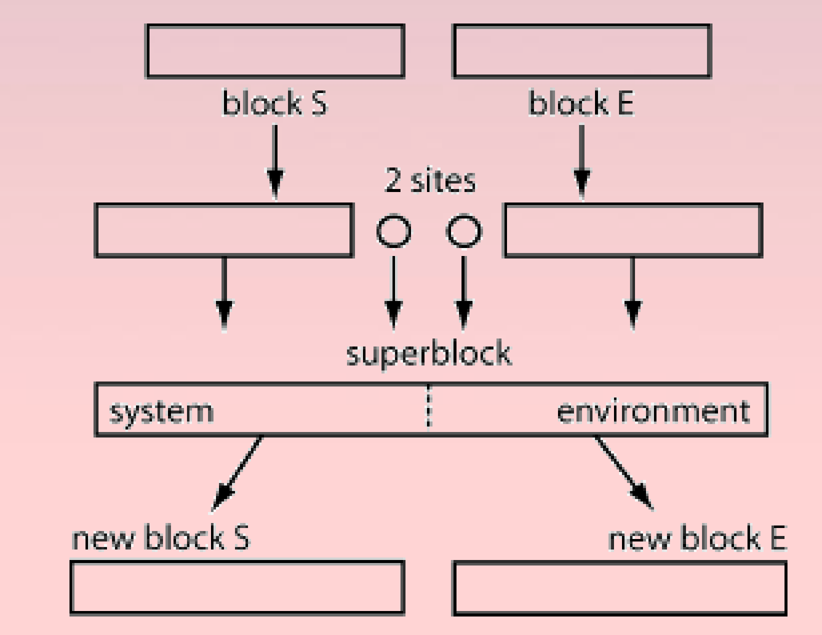
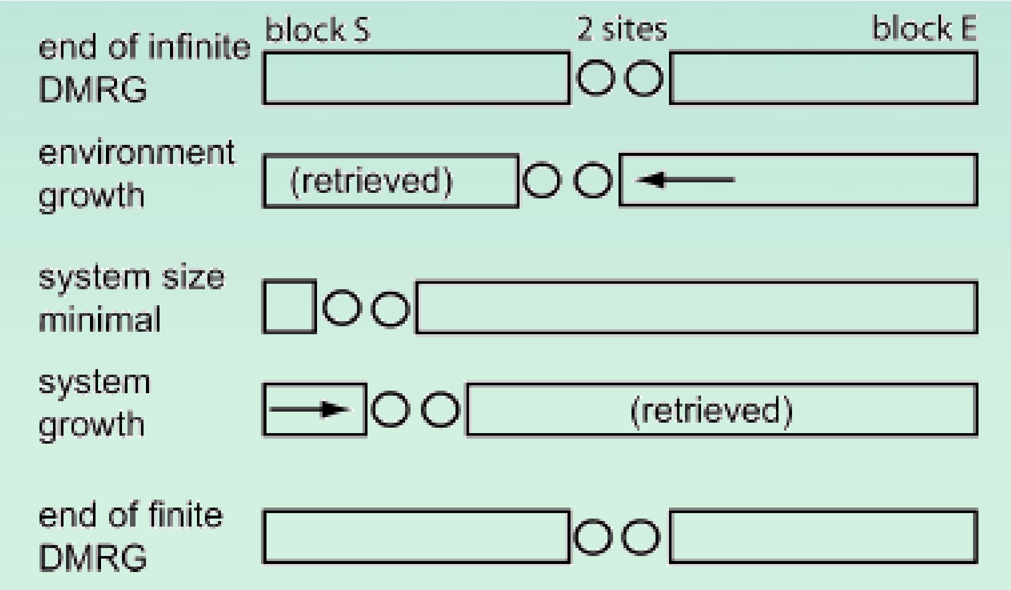
The gold standard: DMRG

Exponential growth of Hilbert space avoided by truncating using local information while keeping non-trival quantumness of states

- Infinite and finite-system algorithms





[S. R. White, Phys. Rev. Lett. 69, 2863 (1992); U. Schollwöck, Ann. Phys. 326, 96 (2011).

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MPS, MPO formulation of DMRG

- DMRG builds up a matrix-product state

$$
\begin{aligned}
|\Psi\rangle &= \sum_{\sigma_1,\ldots,\sigma_L} C_{\sigma_1,\ldots,\sigma_L} |\sigma_1,\ldots,\sigma_L\rangle \\
&\approx \sum_{\sigma_1,\ldots,\sigma_L} \sum_{a_1,\ldots,a_L} M^{\sigma_1}_{a_1} M^{\sigma_2}_{a_1,a_2} \cdots M^{\sigma_{L-1}}_{a_{L-2},a_{L-1}} M^{\sigma_L}_{a_{L-1}} |\sigma_1,\ldots,\sigma_L\rangle
\end{aligned}
$$

- Physical indices (spin) $\sigma_1,\ldots,\sigma_L$
- Tensor $C_{\sigma_1,\ldots,\sigma_L}$ can be decomposed into tensor network.

[S. Ostlund and S. Rommer, Phys. Rev. Lett. 75, 3537 (1995); U. Schollwöck, Ann. Phys. 326, 96 (2011)]

MPS, MPO formulation of DMRG

- Operators are written similarly for consistency

$$O = \sum_{\sigma_1,\ldots,\sigma_L} \sum_{\tau_1,\ldots,\tau_L} D_{\sigma_1,\tau_1\cdots\sigma_L,\tau_L} \left|\sigma_1,\ldots,\sigma_L\right\rangle\left\langle\tau_1,\ldots,\tau_L\right|$$

$$= \sum_{\sigma_1,\ldots,\sigma_L} \sum_{\tau_1,\ldots,\tau_L} \sum_{b_1,\ldots,b_L} W_{b_1}^{\sigma_1,\tau_1} W_{b_1,b_2}^{\sigma_2,\tau_2} \cdots W_{b_{L-2},b_{L-1}}^{\sigma_{L-1},\tau_{L-1}} W_{b_{L-1}}^{\sigma_L,\tau_L} \times$$

$$\left|\sigma_1,\ldots,\sigma_L\right\rangle\left\langle\tau_1,\ldots,\tau_L\right|$$

[S. Ostlund and S. Rommer, Phys. Rev. Lett. 75, 3537 (1995); U. Schollwöck, Ann. Phys. 326, 96 (2011)]

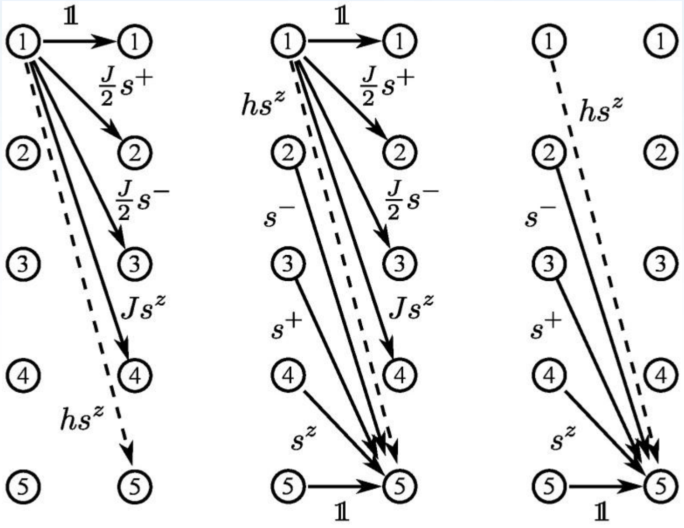
MPOs in practice - an example:

$$W_{b_{i-1},b_i} = \begin{pmatrix} 1 & \frac{J_i}{2}s_i^+ & \frac{J_i}{2}s_i^- & J_i s_i^z & 0 \\ 0 & 0 & 0 & 0 & s_i^- \\ 0 & 0 & 0 & 0 & s_i^+ \\ 0 & 0 & 0 & 0 & s_i^z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$W_{b_2,b_3} W_{b_3,b_4} =$$

$$= \begin{pmatrix} 1 & \frac{J_3}{2}s_3^+ & \frac{J_3}{2}s_3^- & J_3 s_3^z & 0 \\ 0 & 0 & 0 & 0 & s_3^- \\ 0 & 0 & 0 & 0 & s_3^+ \\ 0 & 0 & 0 & 0 & s_3^z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{J_4}{2}s_4^+ & \frac{J_4}{2}s_4^- & J_4 s_4^z & 0 \\ 0 & 0 & 0 & 0 & s_4^- \\ 0 & 0 & 0 & 0 & s_4^+ \\ 0 & 0 & 0 & 0 & s_4^z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{J_4}{2}s_4^+ & \frac{J_4}{2}s_4^- & J_4 s_4^z & \frac{J_3}{2}s_3^+ s_4^- + \frac{J_3}{2}s_3^- s_4^+ + J_3 s_3^z s_4^z \\ 0 & 0 & 0 & 0 & s_4^- \\ 0 & 0 & 0 & 0 & s_4^+ \\ 0 & 0 & 0 & 0 & s_4^z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



[S. Ostlund and S. Rommer, Phys. Rev. Lett. 75, 3537 (1995); U. Schollwöck, Ann. Phys. 326, 96 (2011)]

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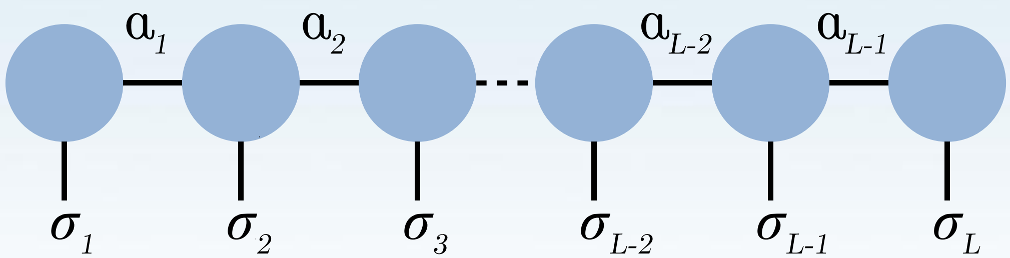
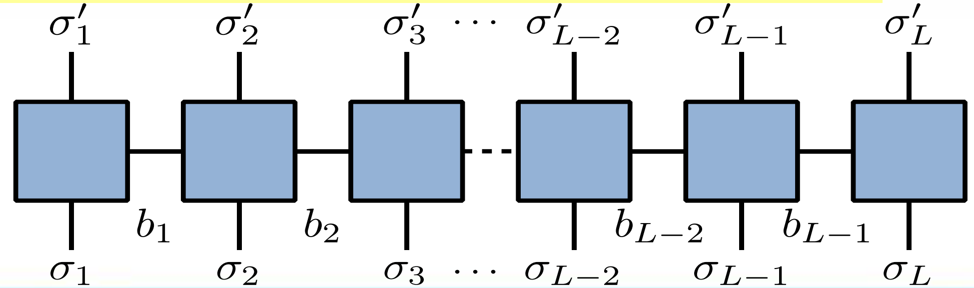
MPS, MPO formulation of DMRG

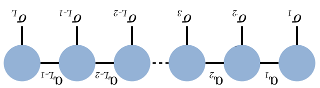
- Moving to **cartoons**

$$|\Psi\rangle = \sum_{\sigma_1,\ldots,\sigma_L} \sum_{a_1,\ldots,a_L} M^{\sigma_1}_{a_1} M^{\sigma_2}_{a_1,a_2} \cdots M^{\sigma_{L-1}}_{a_{L-2},a_{L-1}} M^{\sigma_L}_{a_{L-1}} |\sigma_1,\ldots,\sigma_L\rangle$$



$$O = \sum_{\sigma_1,\ldots,\sigma_L} \sum_{\tau_1,\ldots,\tau_L} \sum_{b_1,\ldots,b_L} W^{\sigma_1,\tau_1}_{b_1} W^{\sigma_2,\tau_2}_{b_1,b_2} \cdots W^{\sigma_{L-1},\tau_{L-1}}_{b_{L-2},b_{L-1}} W^{\sigma_L,\tau_L}_{b_{L-1}} |\sigma_1,\ldots,\sigma_L\rangle\langle\tau_1,\ldots,\tau_L|$$



$\langle\Psi|$ 

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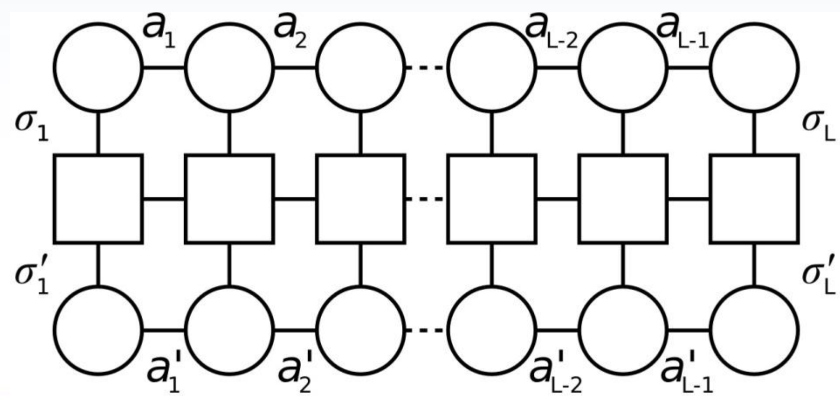
$$O = \sum_{\sigma_1,\ldots,\sigma_L} \sum_{\tau_1,\ldots,\tau_L} \sum_{b_1,\ldots,b_L} W^{\sigma_1,\tau_1}_{b_1} W^{\sigma_2,\tau_2}_{b_1,b_2} \cdots W^{\sigma_{L-1},\tau_{L-1}}_{b_{L-2},b_{L-1}} W^{\sigma_L,\tau_L}_{b_{L-1}} |\sigma_1,\ldots,\sigma_L\rangle\langle\tau_1,\ldots,\tau_L|$$

$$\langle\Psi|H|\Psi\rangle =$$

$$\sum_{\sigma_1,\ldots,\sigma_L} \sum_{\sigma_1',\ldots,\sigma_L'} \sum_{a_1,\ldots,a_L} \sum_{a_1',\ldots,a_L'} \sum_{b_1,\ldots,b_L} M^{*\sigma_1}_{a_1} W^{\sigma_1,\sigma_1'}_{1,b_1} M^{\sigma_1'}_{1,a_1'} M^{*\sigma_2}_{a_1,a_2} W^{\sigma_2,\sigma_2'}_{b_1,b_2} M^{\sigma_2'}_{a_1',a_2'} \cdots$$

$$\cdots M^{*\sigma_{L-1}}_{a_{L-2},a_{L-1}} W^{\sigma_{L-1},\sigma_{L-1}}_{b_{L-2},b_{L-1}} M^{\sigma_{L-1}'}_{a_{L-2}',a_{L-1}'} M^{*\sigma_L}_{a_{L-1},1} W^{\sigma_L,\sigma_L}_{b_{L-1}} M^{a_L'}_{a_{L-1}'}$$



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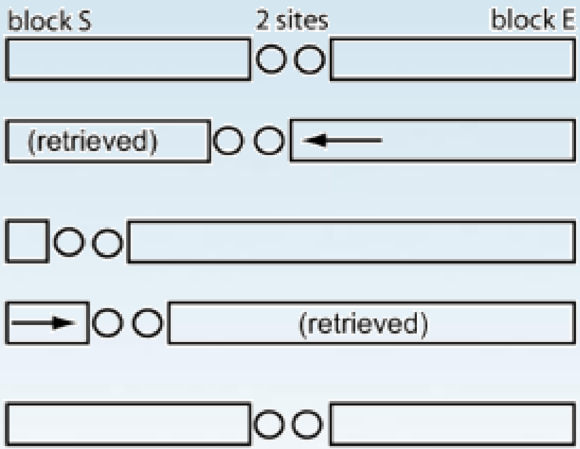
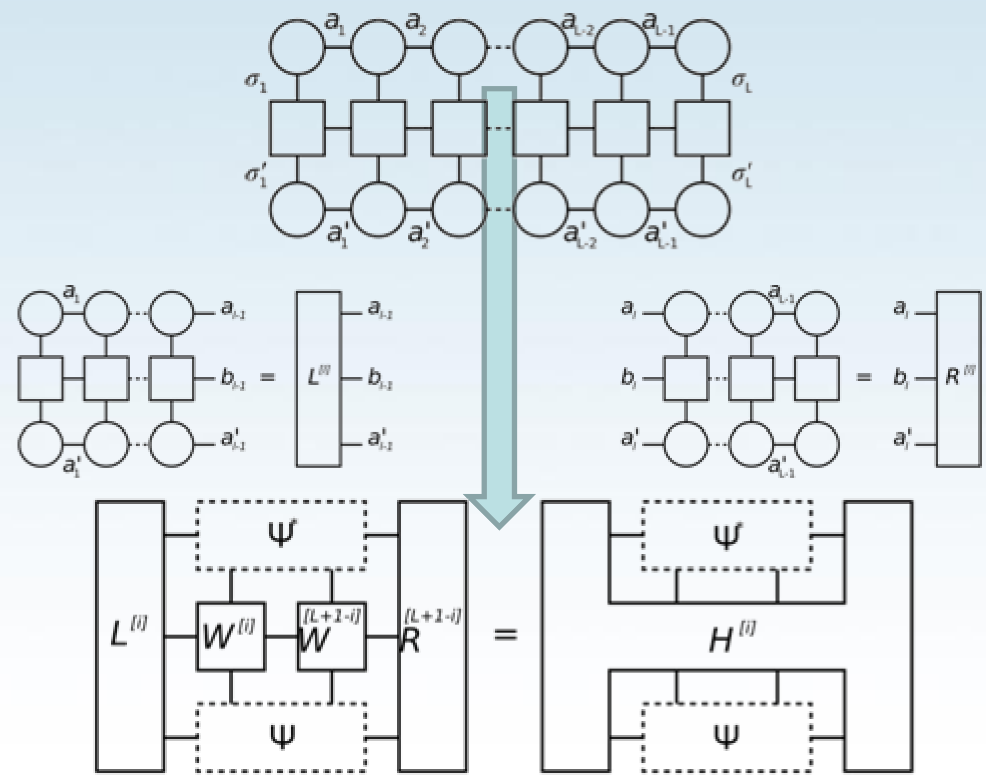
MPS, MPO formulation of DMRG

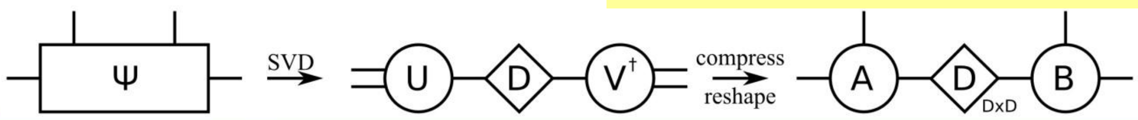




- Reformulation of 2-site DMRG
- Density matrix Schur-decomposed and truncated

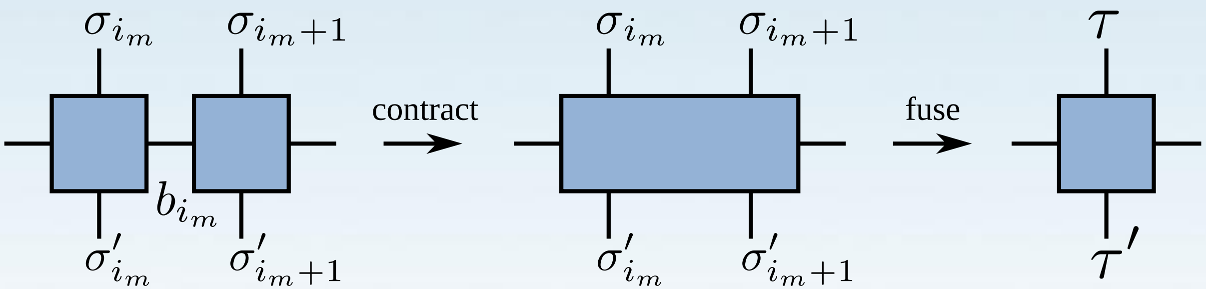
$A = UDV^{\dagger}$

$\chi = \text{bond dimension}$

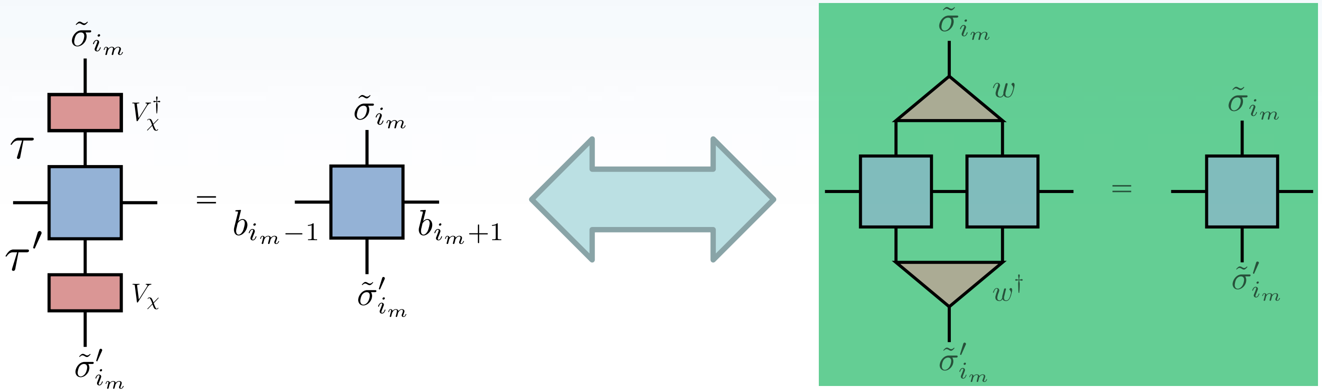


SDRG MPO process

- Contract MPO tensors for sites with largest gap



- Keep lowest χ eigenvalues only and contract

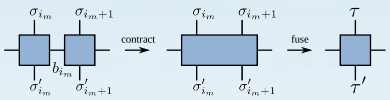


A. M. Goldsborough, RAR, Phys. Rev. B **89**, **214203-11 (2014)**

SDRG MPO process

- Contract MPO tensors for sites with largest gap

$$W^{[i_m,i_{m+1}]} = \sum_{b_{i_m}} W^{\sigma_{i_m},\sigma_{i_m}'}_{b_{i_{m-1}},b_{i_m}} W^{\sigma_{i_{m+1}},\sigma_{i_{m+1}}'}_{b_{i_m},b_{i_{m+1}}}$$



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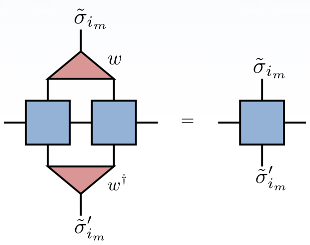


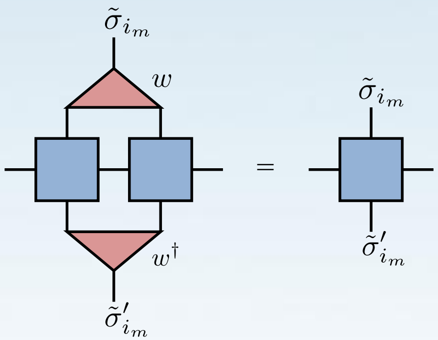
$$\Delta_\chi = V_\chi^\dagger \left(H^B_{i_m} \otimes 1 + J_{i_m} \vec{s}^{\,R}_{i_m} \bullet \vec{s}^{\,L}_{i_{m+1}} + 1 \otimes H^B_{i_m+1} \right) V_\chi$$

$$W^{\tilde{\sigma}_{i_m},\tilde{\sigma}_{i_m}'}_{b_{i_m-1},b_{i_m}} = \sum_{\tau,\tau'} \left[V_\chi^\dagger \right]^{\tilde{\sigma}_{i_m}}_{\tau} W^{\tau,\tau'}_{b_{i_m-1},b_{i_m}} \left[V_\chi \right]^{\tilde{\sigma}_{i_m}'}_{\tau'}$$

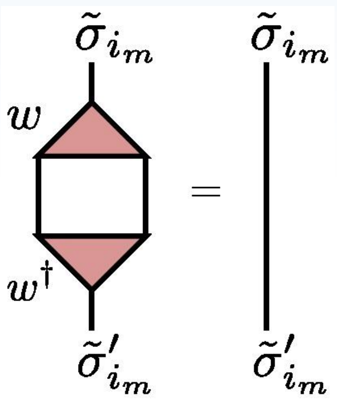
A. M. Goldsborough, RAR, Phys. Rev. B **89**, 214203-11 (2014)

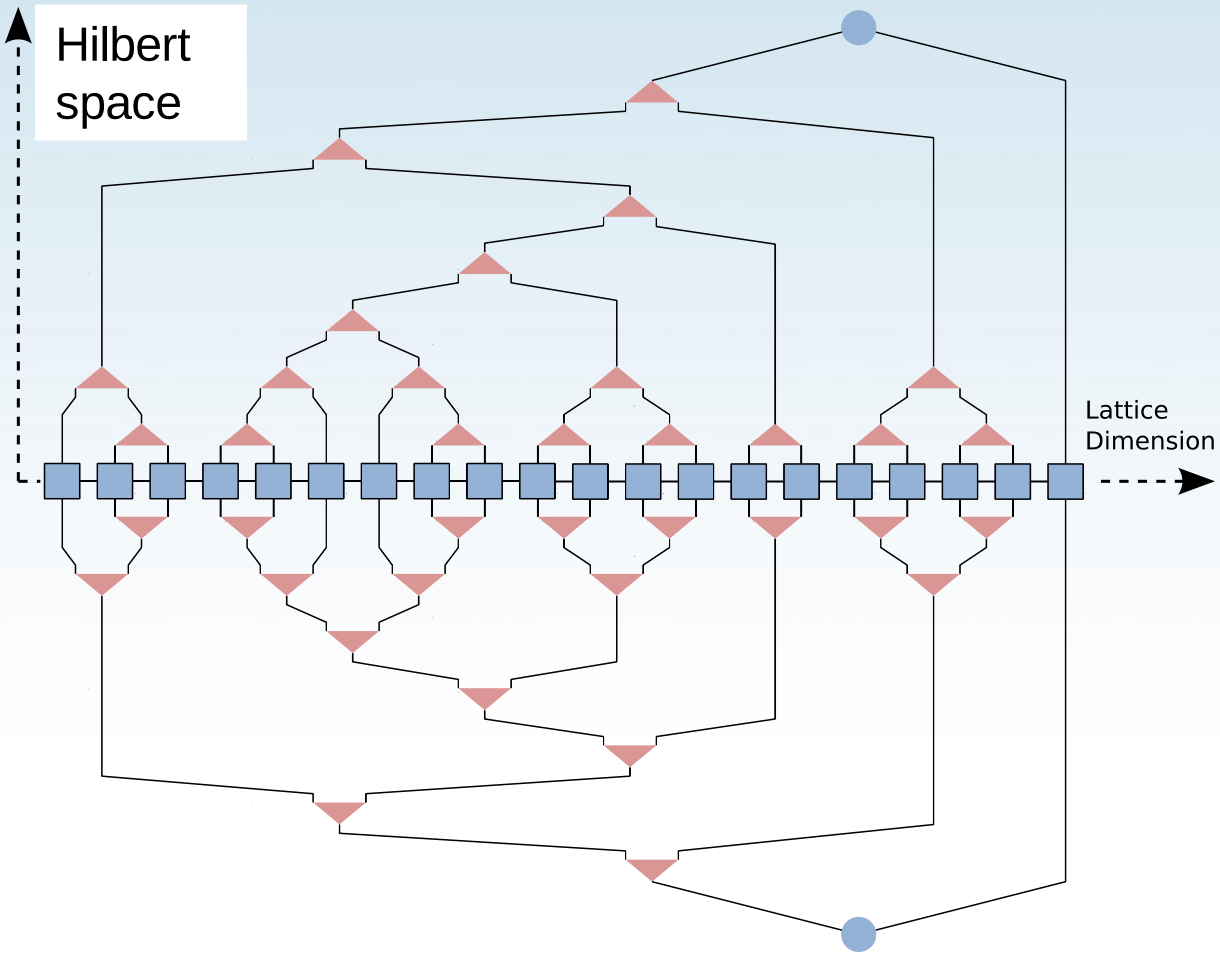
SDRG as TTN

Basic RG step:

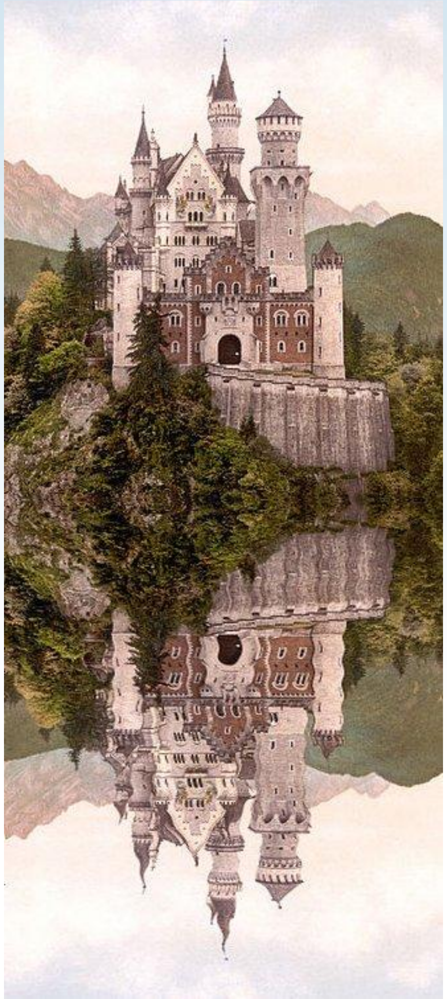


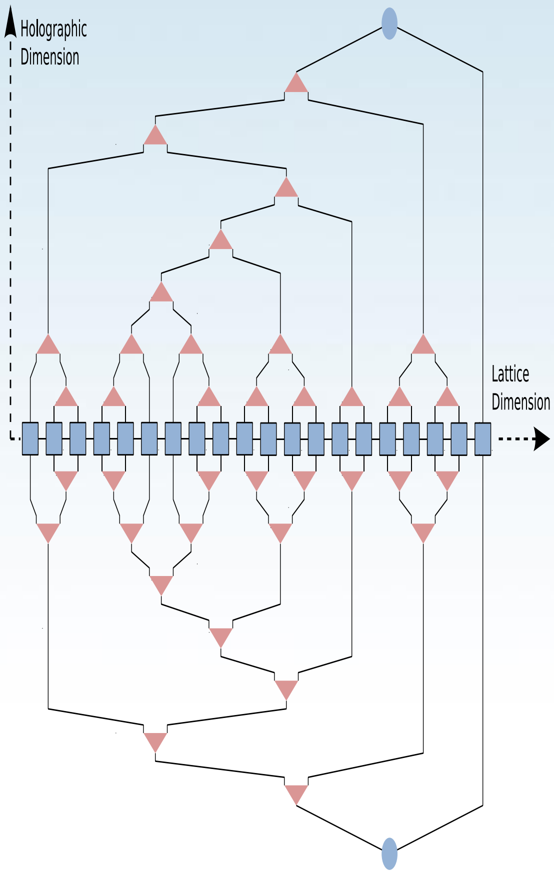
Isometry:





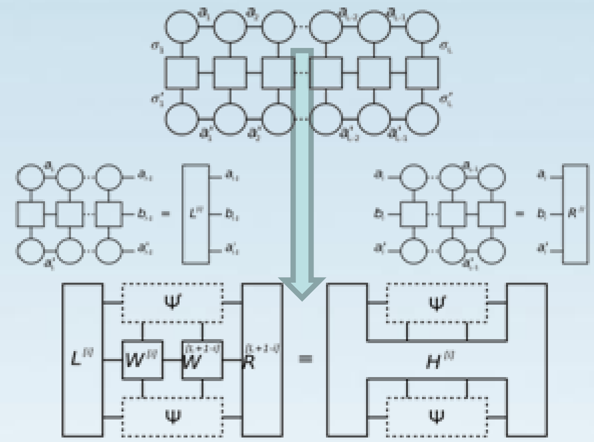
Tree tensor networks pictorially

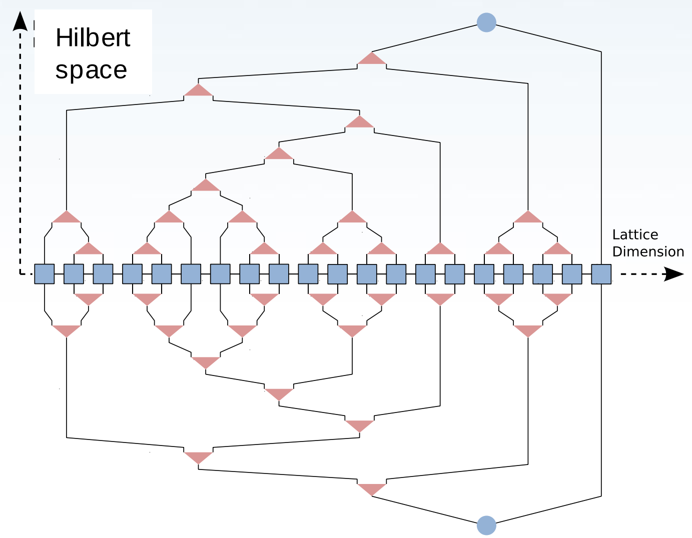




Why bother?

- vMPS works well for clean systems
- vMPS ignores disorder in setup and hence update process is highly affected
- Inhomogeneous TTN as presented here includes disorder as building block of network itself, ADAPTIVE





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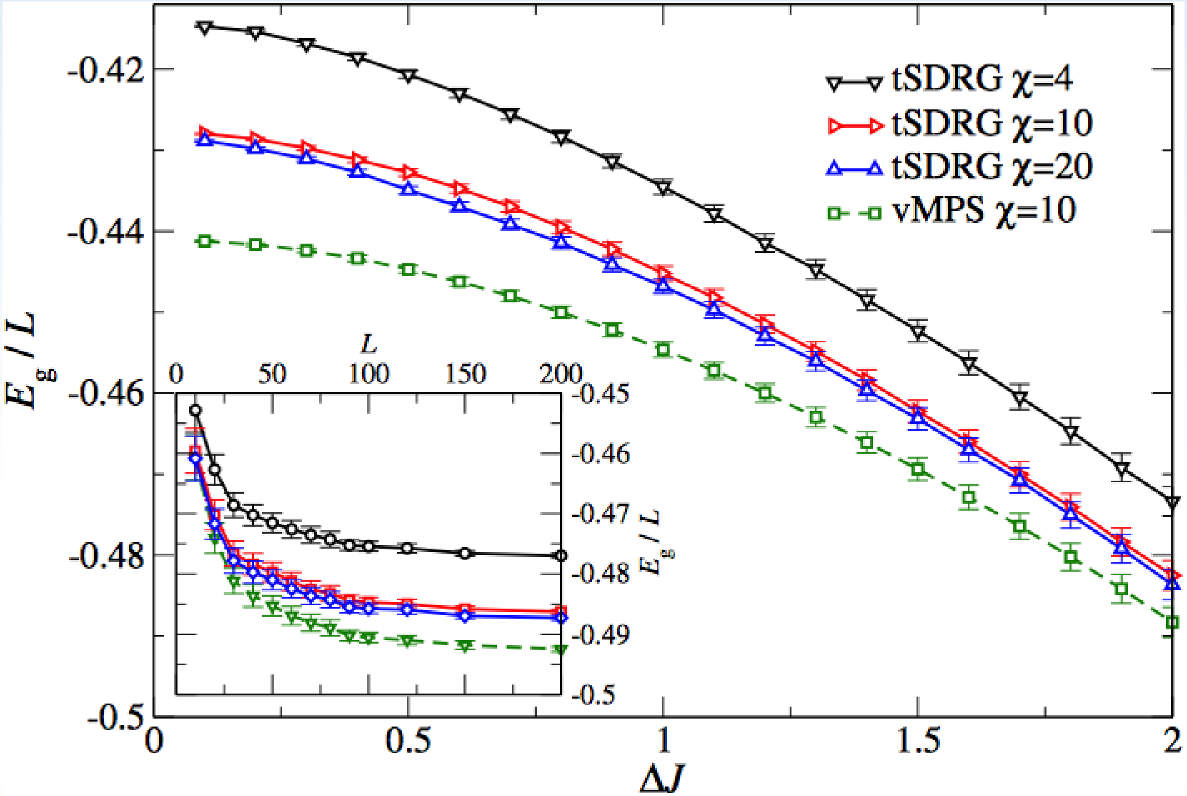
Ground state convergence

DMRG=**vMPS**

SDRG with TTN=**tSDRG**

$$J_i \in \left[1-\frac{\Delta J}{2}, 1+\frac{\Delta J}{2}\right]$$

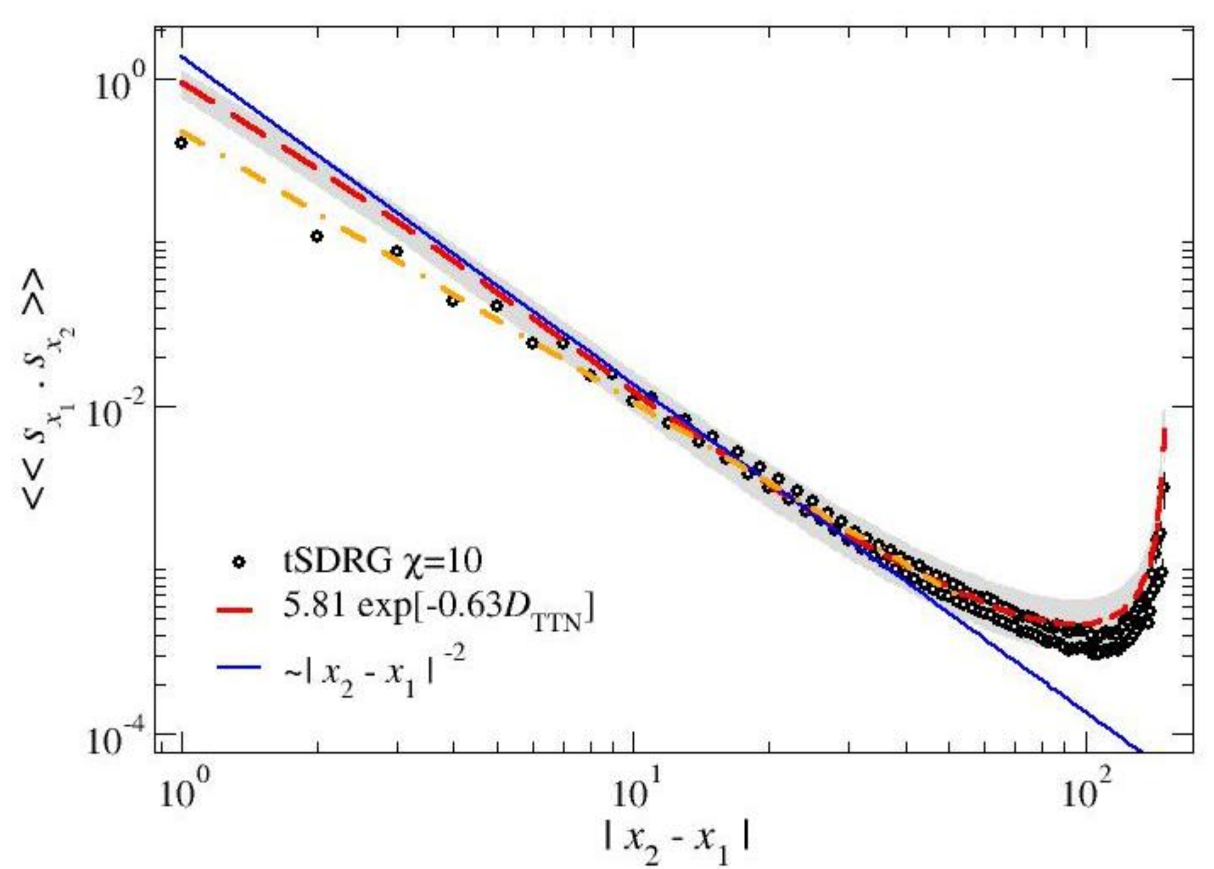
- tSRDG converges
- Energies a bit better in **v**MPS
- Larger disorder improves tSDRG



Spin-spin correlation

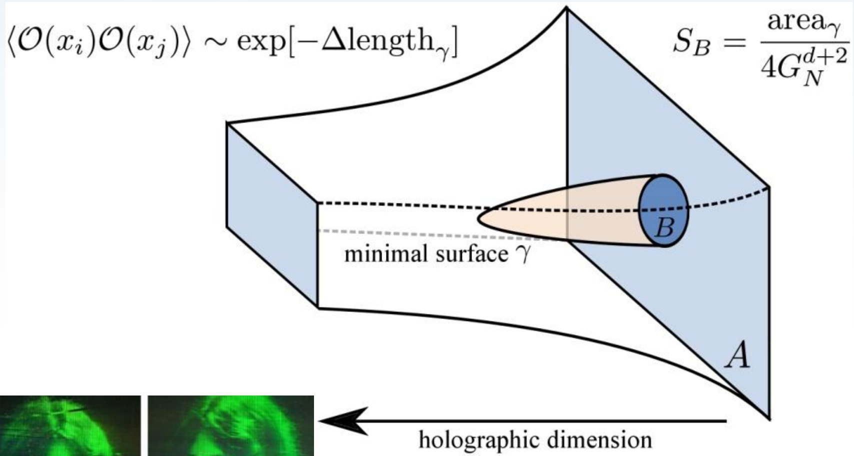
$$\overline{\left\langle \vec{s}_i \bullet \vec{s}_j \right\rangle} \propto \frac{1}{|i-j|^2}$$

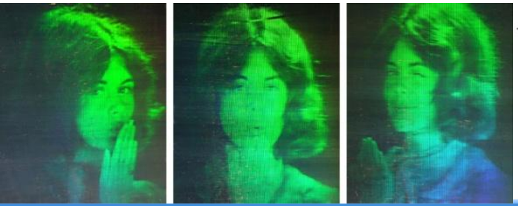
- Fisher 1/r^2 recovered
- Large distance behaviour dominated by boundaries



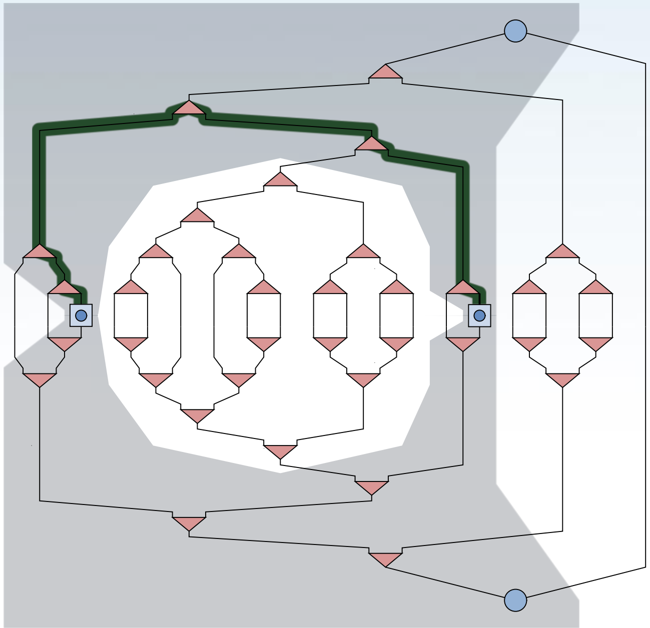
The idea of "holography"

- Maldacena/Witten: physical world lives on surface of larger/bulk space
- Swingle: tensor networks are like larger space





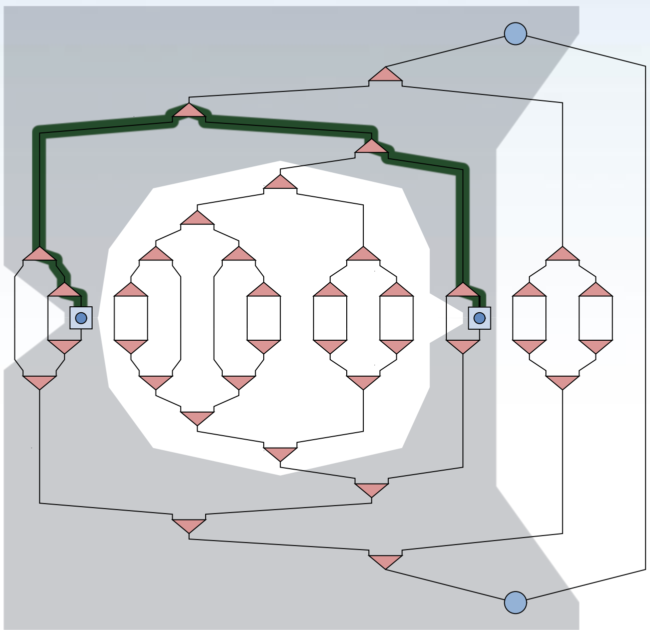
- Evenbly/Vidal: might help in computing correlations
  - Causal cone

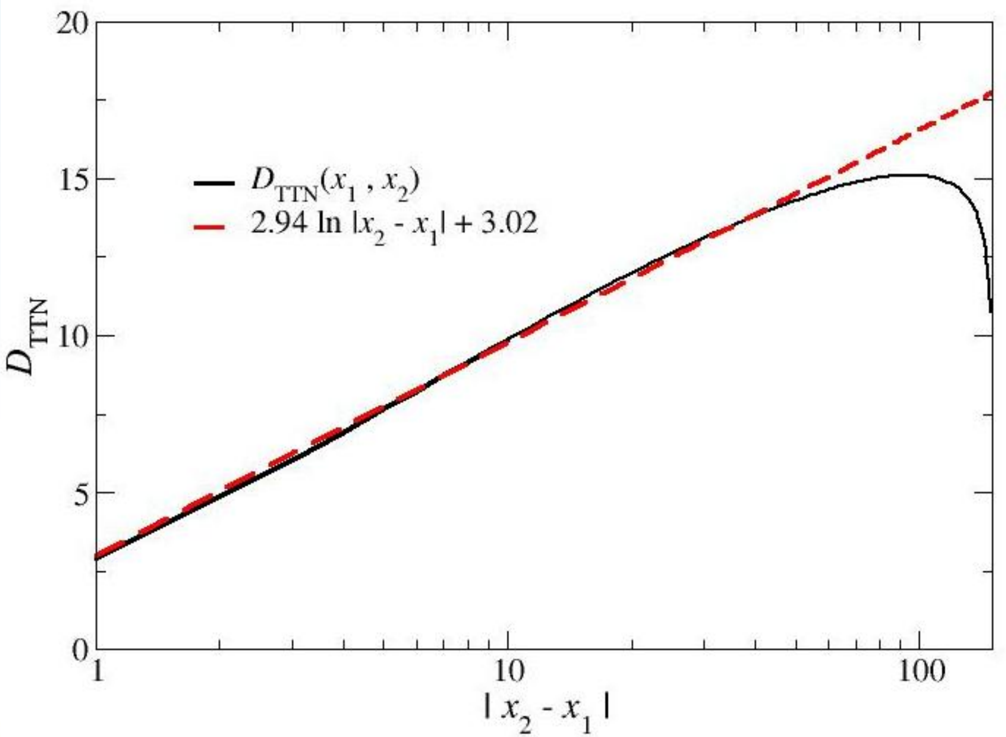


[J. Maldacena, Int. J. Theor. Phys. 38, 1113 (1999); B. Swingle, Phys. Rev. D 86, 065007 (2012); J. Molina-Vilaplana, J. High Energ. Phys. 2013, 1 (2013); S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 181602 (2006); J. McGreevy, Adv. High Energy Phys. 2010, 723105 (2010); G. Evenbly and G. Vidal, J. Stat. Phys. 145, 891 (2011)]
]

Fitting a path length

- Proposal: $\overline{\left\langle \vec{s}_{x_1} \bullet \vec{s}_{x_2} \right\rangle} \propto e^{-\alpha \left\langle D(x_1, x_2) \right\rangle} \propto e^{-\alpha\beta \log|x_1 - x_2|} \propto |x_1 - x_2|^{-a}$





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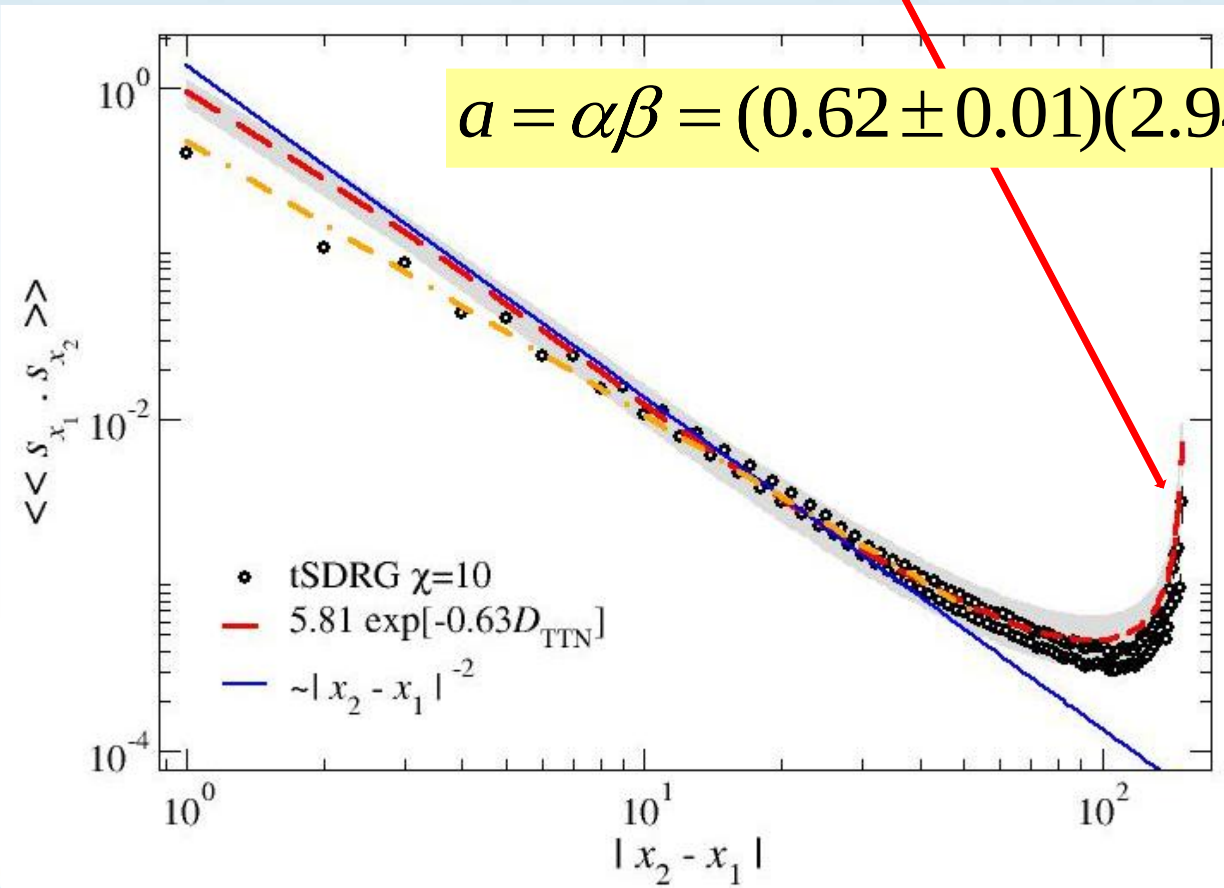
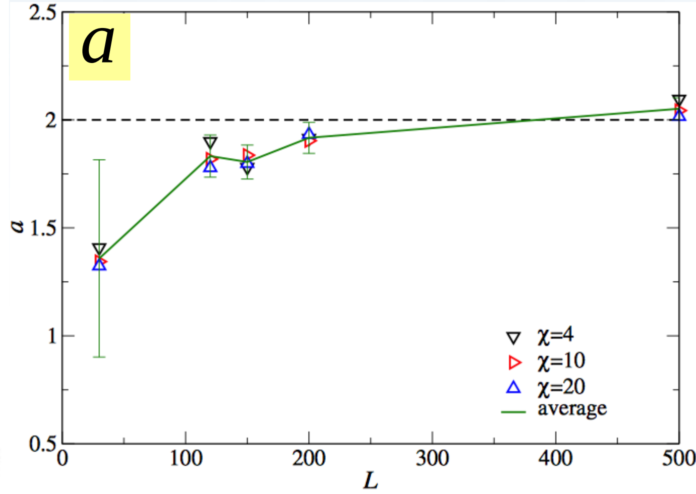
Spin-spin correlation as a **holographic path length**

$$a = \alpha\beta = (0.62 \pm 0.01)(2.94 \pm 0.02) = 1.84 \pm 0.04$$

$$L = 150$$





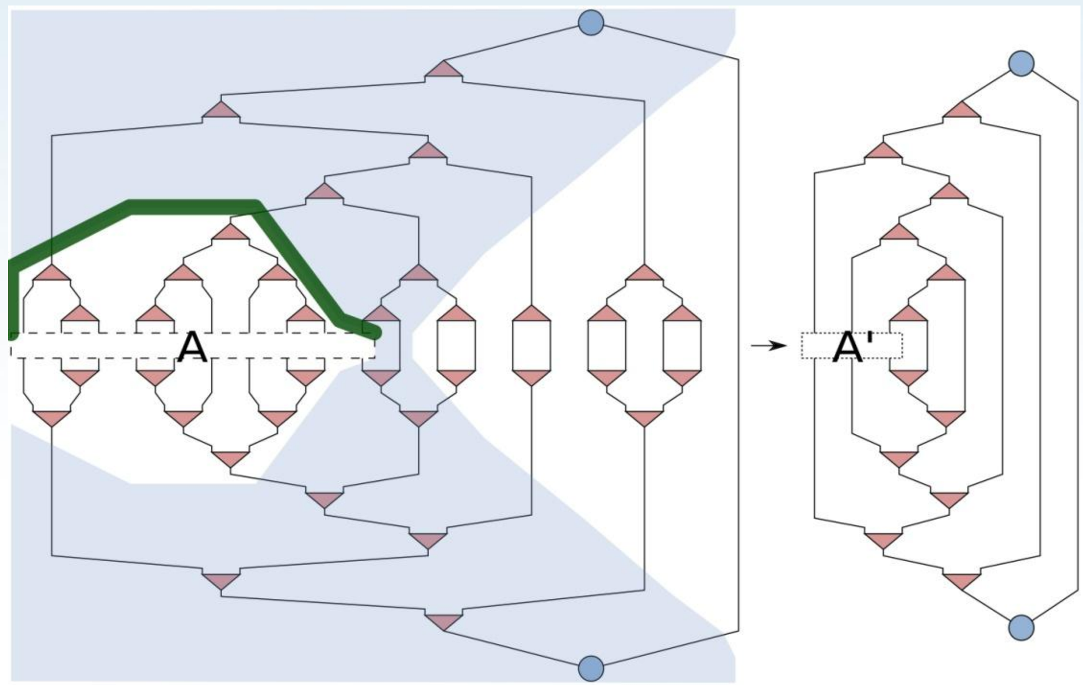
Entanglement entropy

$$S_{A|B} = -\mathrm{Tr}\rho_A \log_2 \rho_A$$

- Holography suggests new method of calculation:

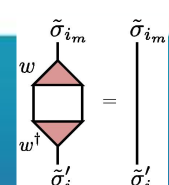
$$S_{A|B} \propto n_A$$

minimum #bonds that need to be “cut” to separate A from B



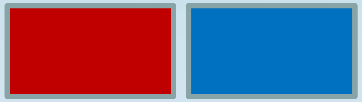
Area law, see [Eisert et al, RMP 82, 277 (2010)]

[idea does not work in vMPS]



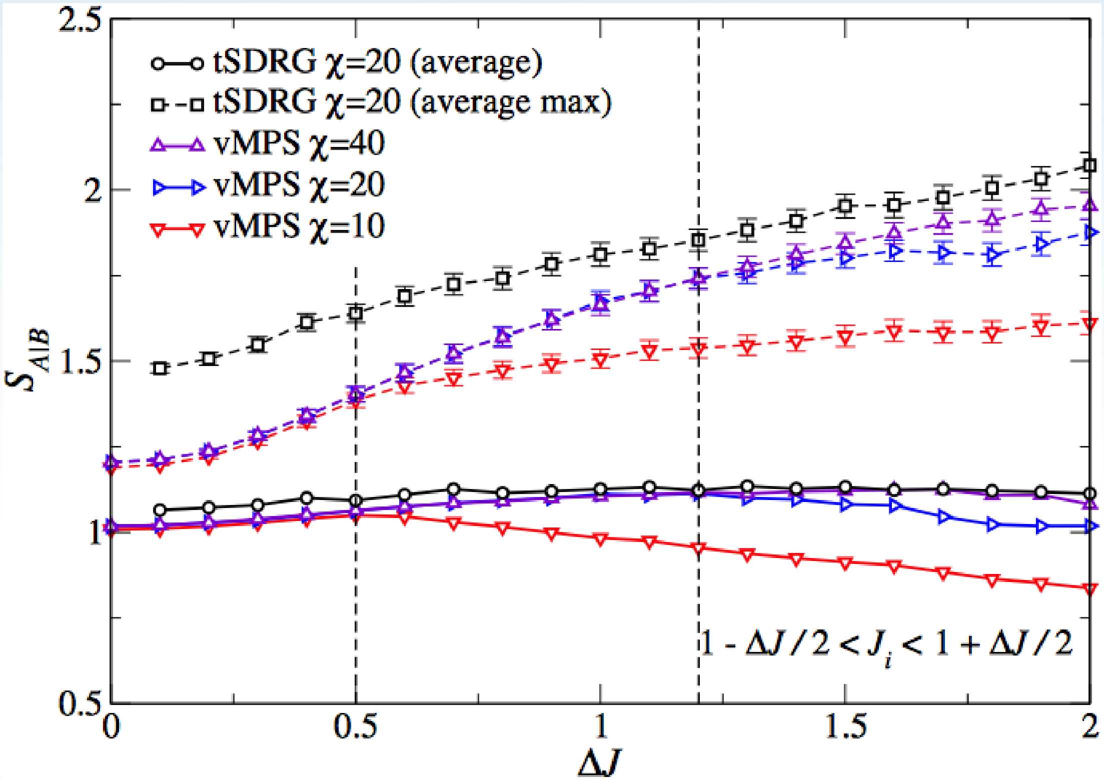
Entanglement entropy

Bipartite system



$$S_{A|B} = -Tr\,\rho_A \log_2 \rho_A$$

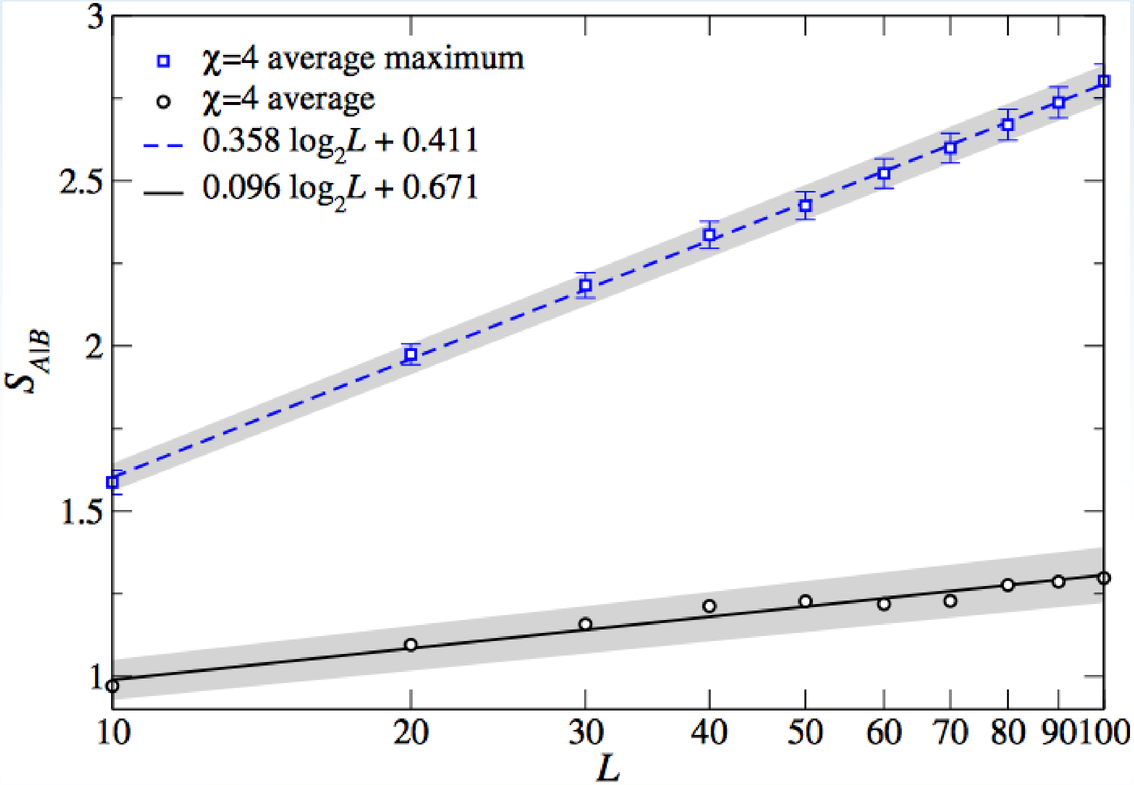
- vMPS best for small ΔJ disorder
- tSDRG better for $\Delta J>1$
- Large difference between mean and maximum S



Log(L) increase for $S_{A|B}$

$\Delta J = 2$

- vMPS needs to increase χ
- tSDRG does not, much easier to compute for large L
- Large difference between mean and maximum *S*



Block entanglement $S_{A,B}$

Block(ed) system

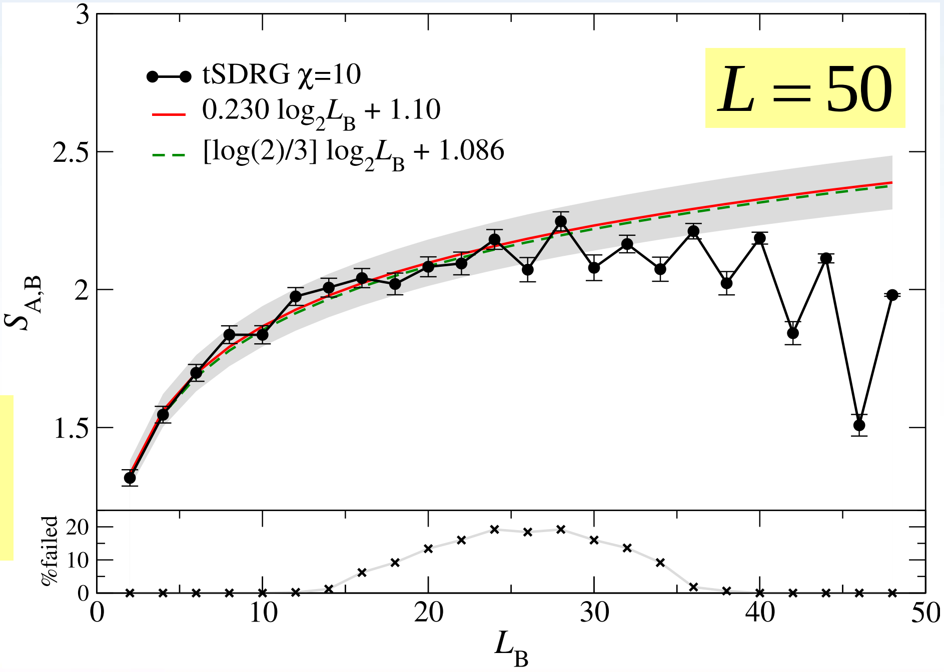
[G. Refael and J. E. Moore, PRL 93, 260602 (2004)]

- Refael/Moore result implies CFT with effective central charge $\tilde{c} = 1 \cdot \log 2$

$$S_{A,B} \sim \frac{\log 2}{3} \log_2 L_B \approx 0.231 \log L_B$$

- We find

$$S_{A,B} \approx 0.230 \log L_B$$

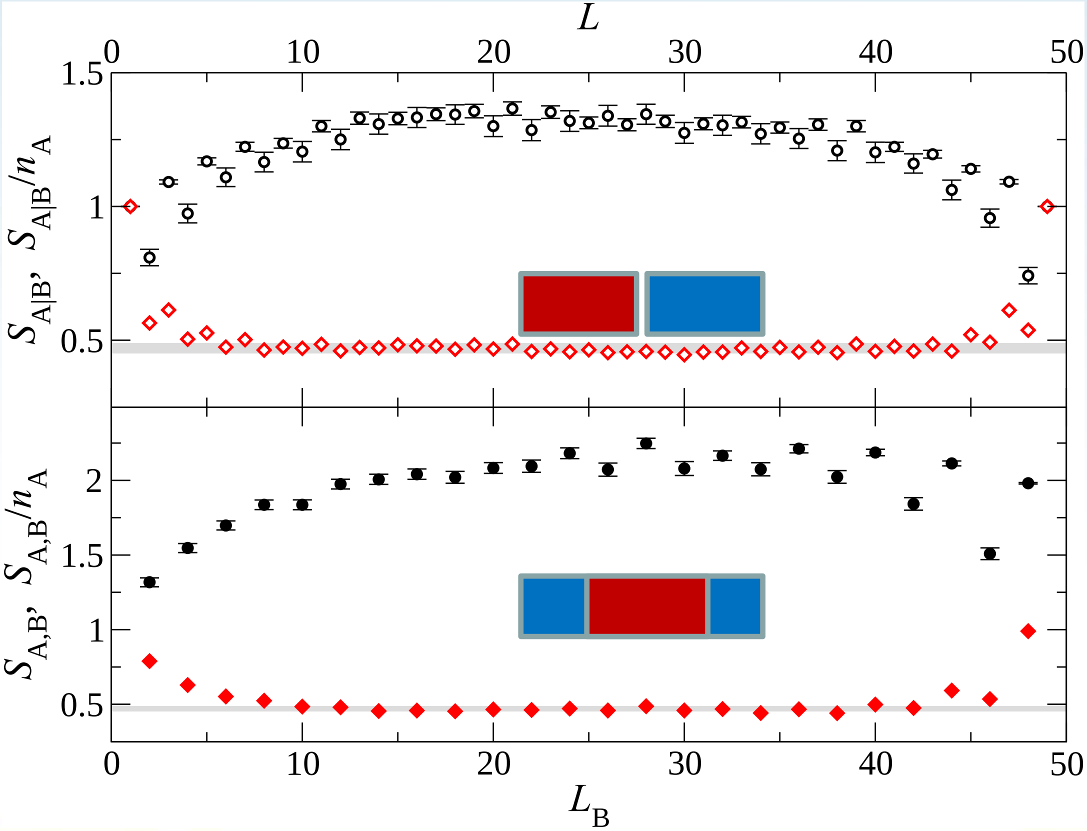


Entanglement per bond, S/n_A

- for $S_{A|B}$ and $S_{A,B}$
- we find constant ratio in bulk

$$\frac{S}{n_A} \approx 0.47 \pm 0.02$$

$L = 50$



- $S/n_A = 1/2$

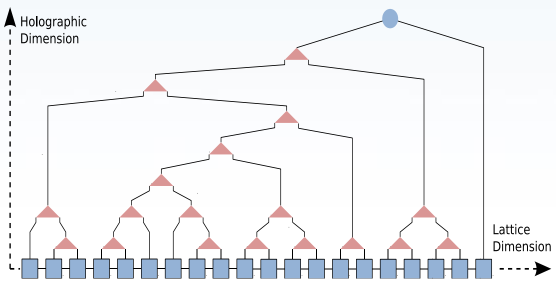
implies

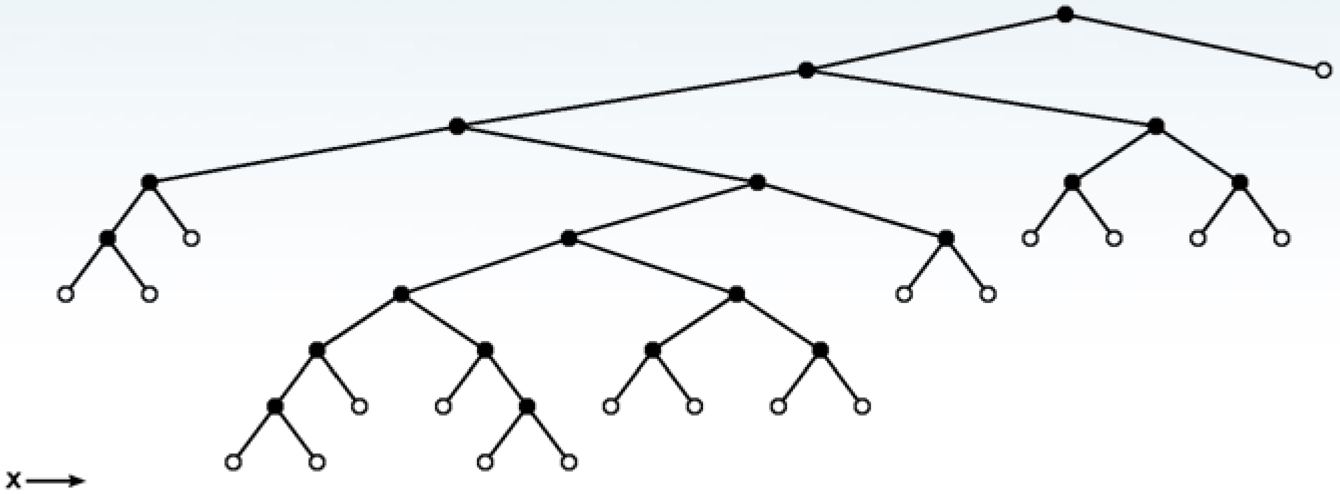
$$n_A = 2\log L_B / 3$$

WARWICK

What about other tree tensor networks?

- Can we perhaps compute the path lengths explicitely for some tree networks?





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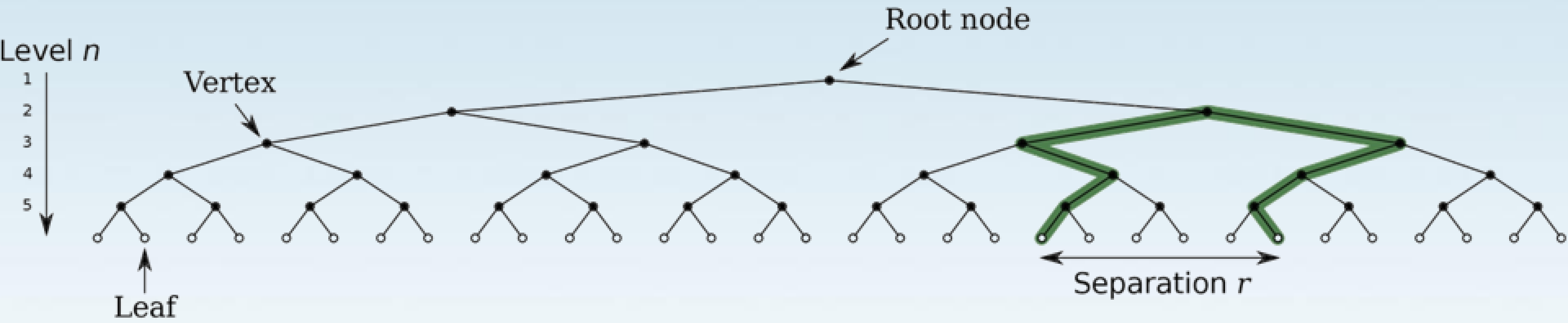
Path lengths in *m*-ary trees



- Full and complete *m*-ary trees have been well studied, e.g. for sorting problems.
- But *1D order* imposes an apparently not yet studied condition.

"Leaf-to-leaf distances and their moments in finite and infinite m-ary tree graphs", **A.M. Goldsborough**, S.A. Rautu, RAR, arXiv:1406.4079

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Path lengths in *m*-ary trees

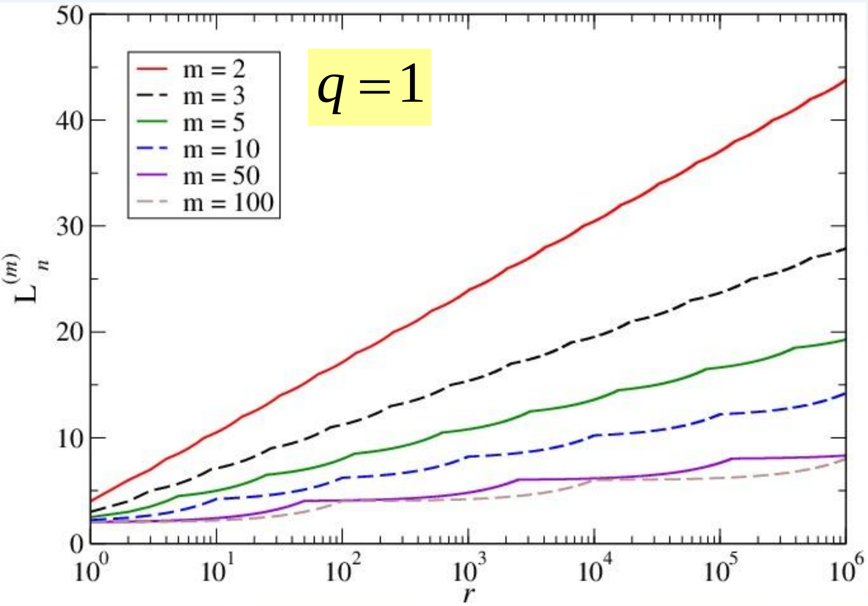
- We find

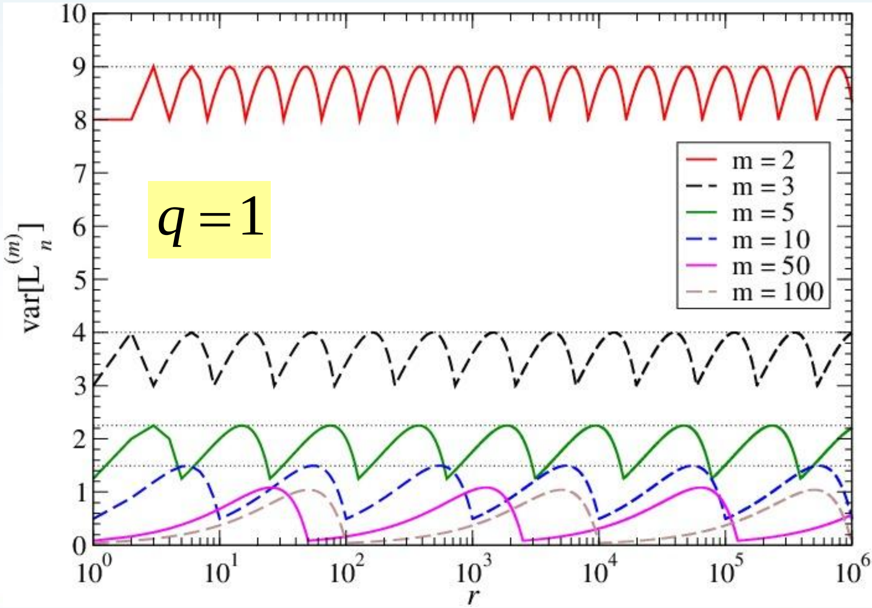
$$\mathbb{L}_{q,n}^{(m)}(r) = \frac{\mathbb{M}_{q,n}^{(m)}(r)}{m^n - r}$$

$$\mathbb{M}_{q,n}^{(m)}(r) = m^{n-n_c} 2^q n_c^q (m^{n_c} - r) + r(m-1)(-2)^q \left[\Phi(m,-q,-n) - m^{n-n_c} \Phi(m,-q,-n_c) \right]$$

$\Phi(m,q,n) =$ Hurwitz-Lerch transcendent

$$n_c^{(m)} = \lfloor \log_m(r) \rfloor + 1$$





"Leaf-to-leaf distances and their moments in finite and infinite m-ary tree graphs", **A.M. Goldsborough**, S.A. Rautu, RAR, arXiv:1406.4079

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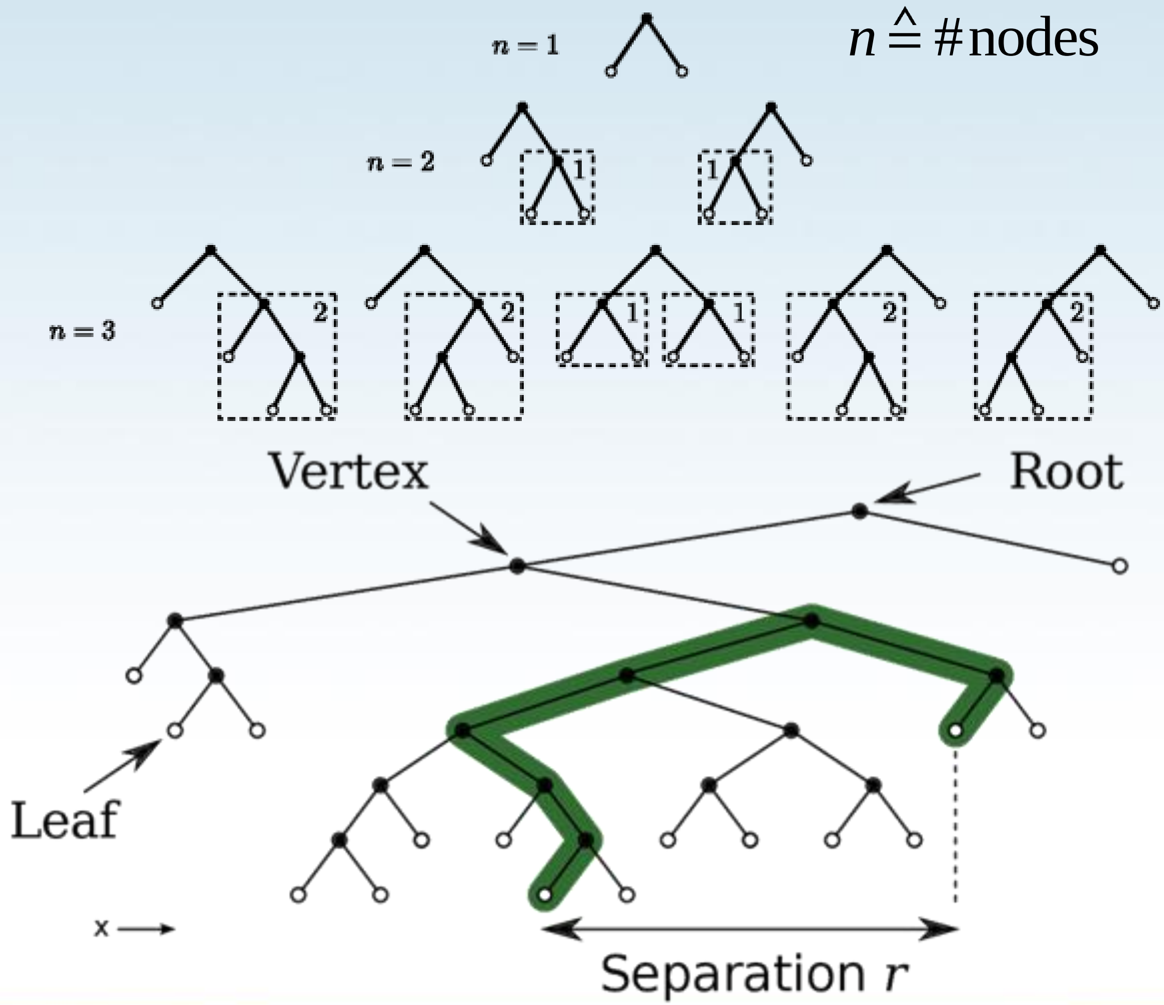
Path lengths in *Catalan* trees

- Full, but not complete
- Catalan numbers 1, 1, 2, 5, 14, ...
- Not a random tree



"Leaf-to-leaf distances in ordered Catalan tree graphs", **A.M. Goldsborough, J.M. Fellows**, M. Bates, S.A. Rautu, G. Rowlands and RAR, soon on arXiv

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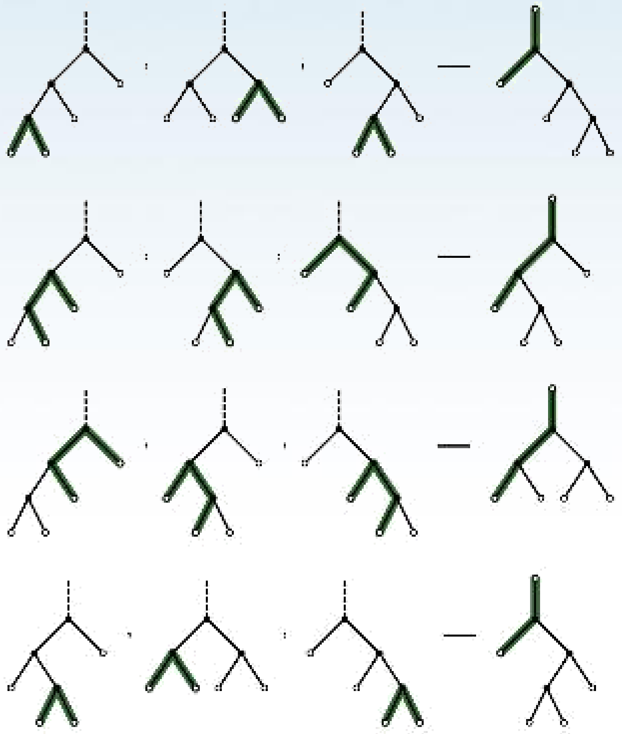
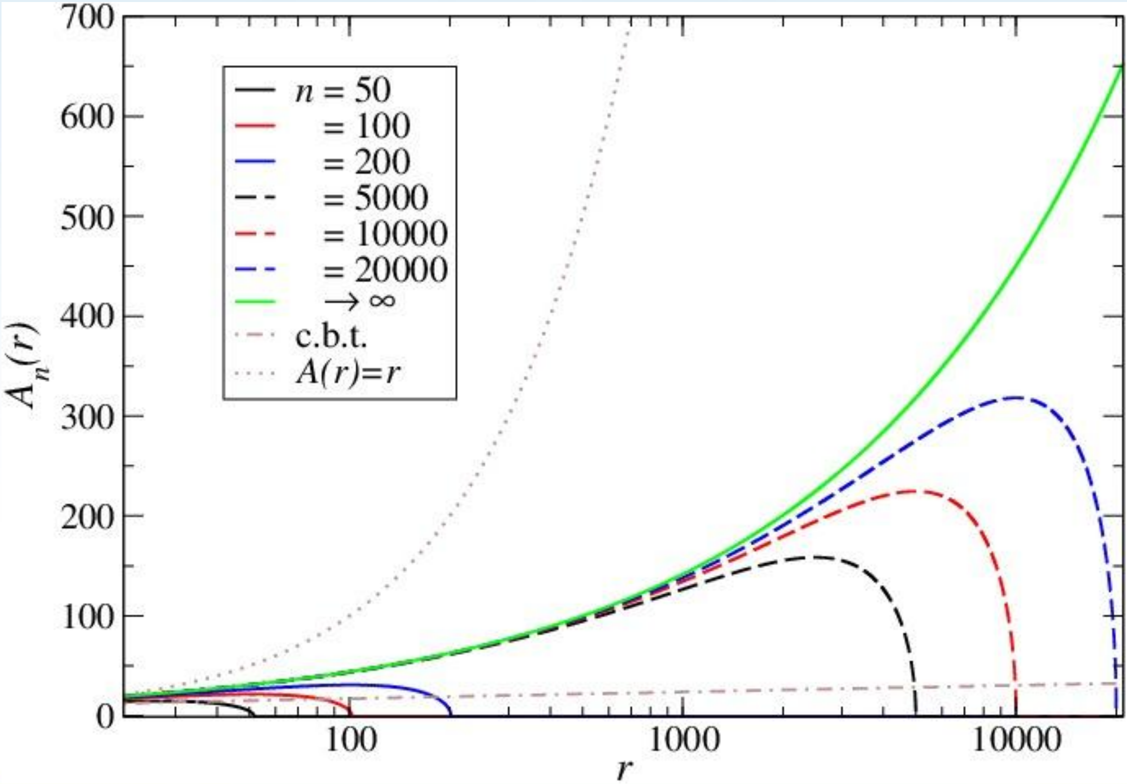
Path lengths in *Catalan* trees

- We find

$$(n-r+1)D_n^r = S_n(r)$$

$$A_n(r) = \frac{D_n^r}{C_n}$$





"Leaf-to-leaf distances in ordered Catalan tree graphs", **A.M. Goldsborough, J.M. Fellows**, M. Bates, S.A. Rautu, G. Rowlands and RAR, soon on arXiv

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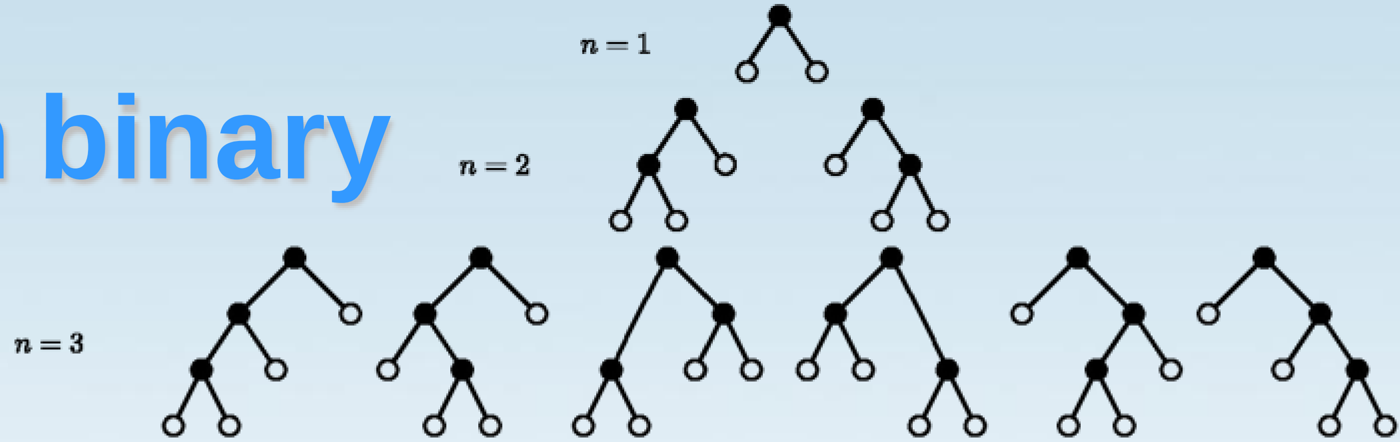
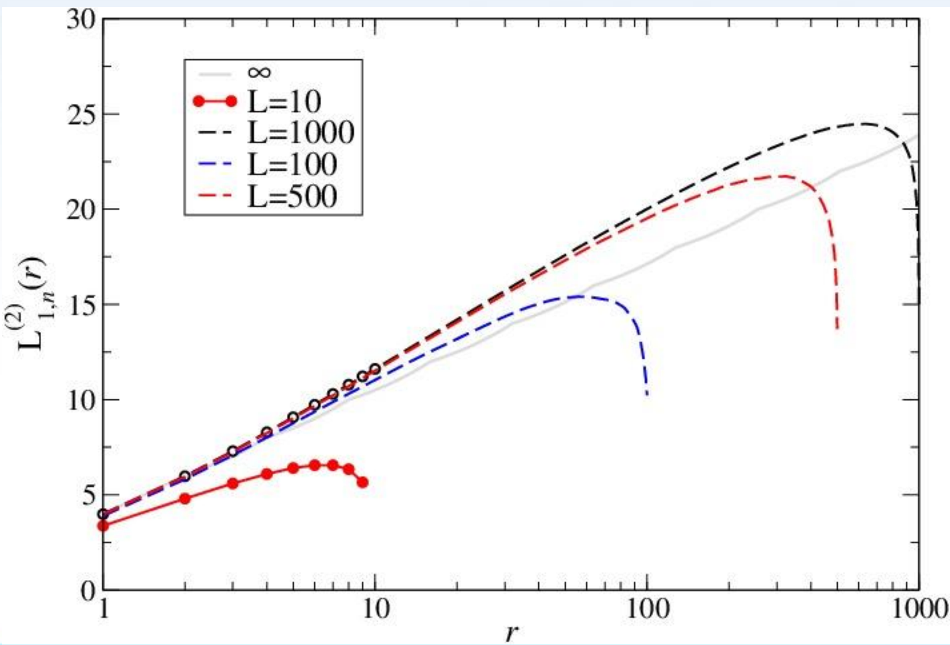
Random binary trees



- not full, nor complete
- n! possibilities
- Computed for 500 samples of size L=10, 100, 500, 1000



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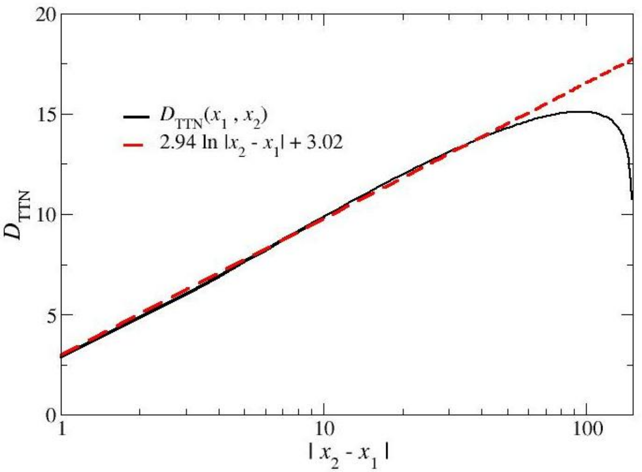
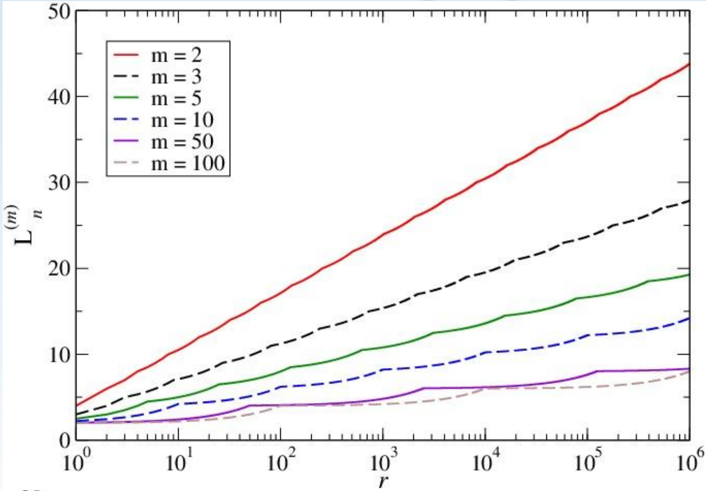
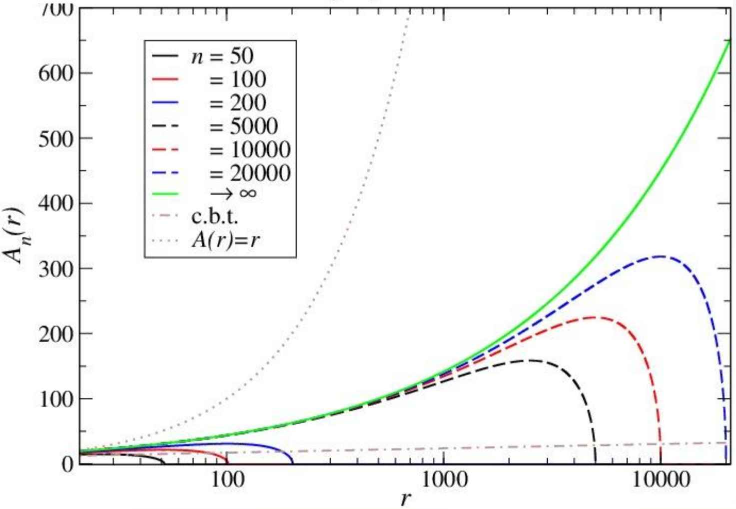
The networks are different, so the physics will be as well … (?)



tSDRG network

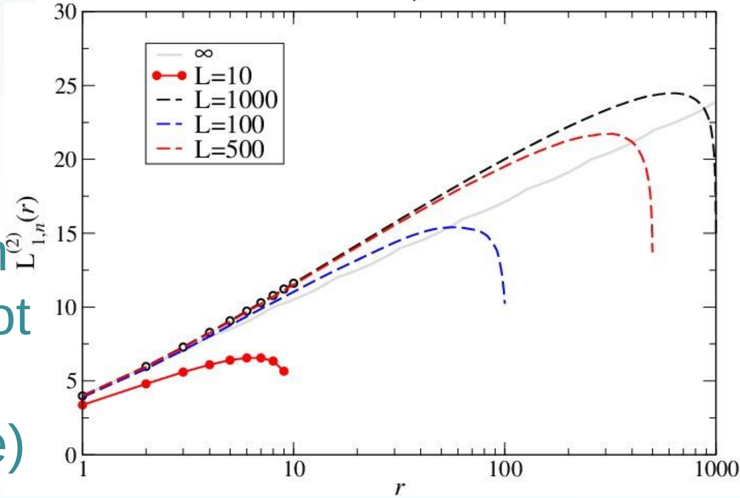
Full and complete *m*-ary trees





Catalan (complete binary) trees

Random binary (not full or complete)



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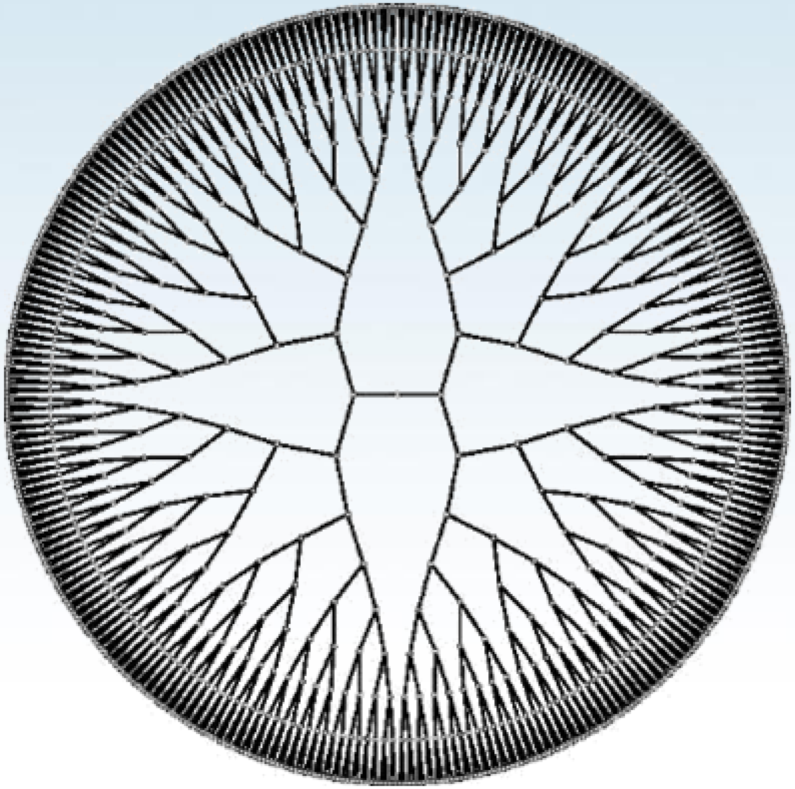
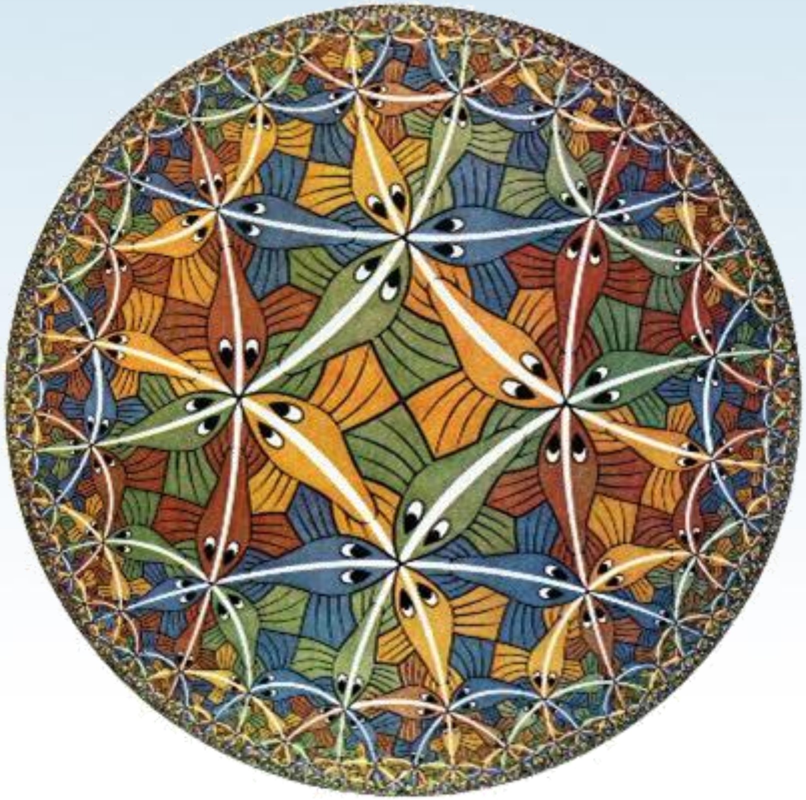
Periodic boundaries (binary)





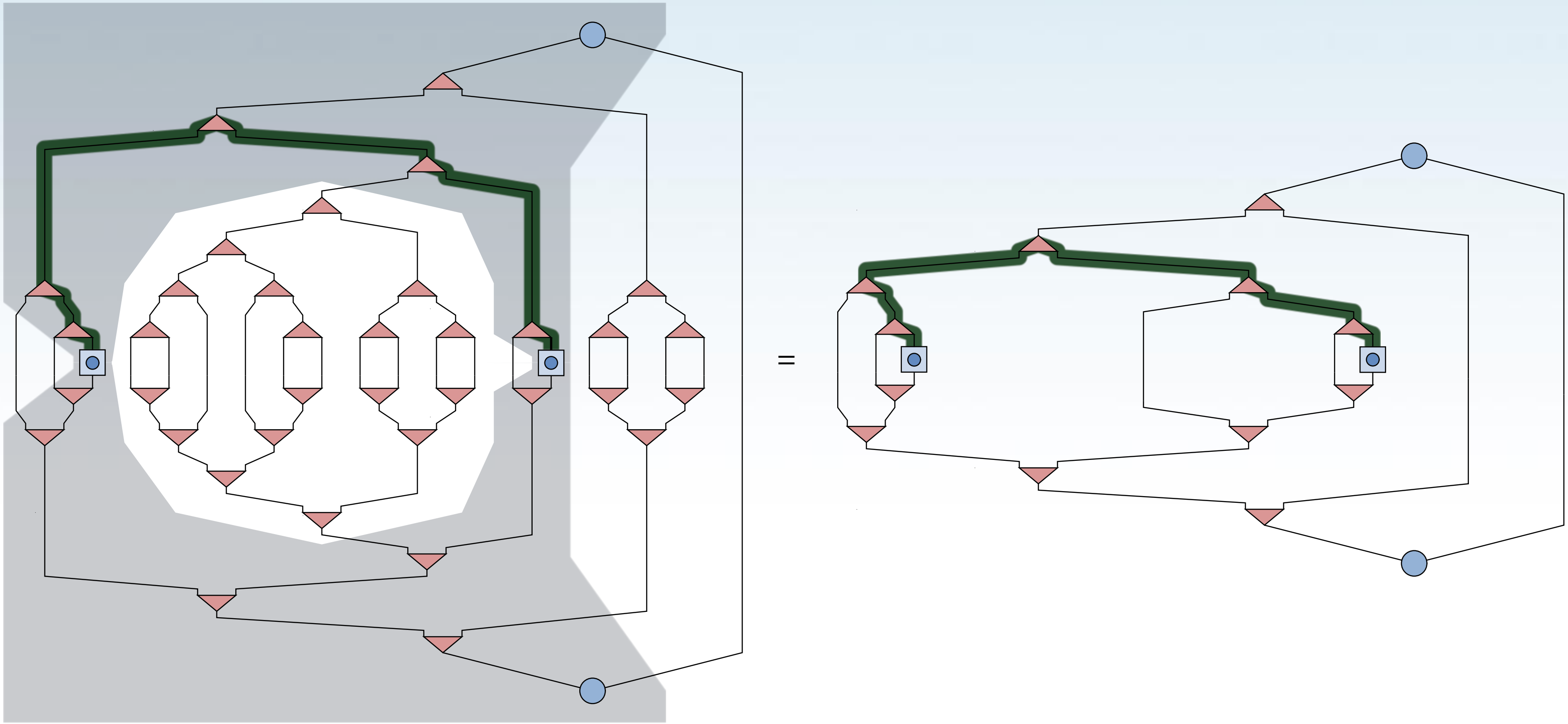
Circle Limit III is a woodcut made in 1959 by Dutch artist M. C. Escher

Conclusions

- An inhomogeneous, self-assembled TTN works. First such beast ever!
- We compared **v**MPS with tSDRG; **v**tSDRG is possible and should be much better (just as finite-size DMRG is better than infinite-size DMRG)
- Holography holds in 1D disordered quantum spin chains, 1st quantitative check!
- Explicit construction of path lengths for a variety of tree graphs: Fun with trees!

A. M. Goldsborough, RAR, Phys. Rev. B **89**, 214203-11 (2014)

Isometries make computation faster



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How to do the RG with a TTN