Parquet equations for interacting and disordered electron systems: Self-energy and the role of the Ward identity

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Microscopic description & macroscopic consistency

- Thermodynamic consistency: from microscopic statistical mechanics to thermodynamic (macroscopic) phenomena
- Macroscopic conservation laws -- microscopic Ward identities
- Baym-Kadanoff construction -- 1P self-consistency for the self-energy
- Quantum critical phenomena -- divergence in two-particle functions
- To control 2P criticality -- a direct (diagrammatic) approach to 2P functions (not via self-energy)
- Parquet equations -- 2P self-consistency eliminating spurious (non-integrable) divergencies

How to make 2P (parquet) approach thermodynamically consistent?

Outline

1. Quantum many-body systems
   - Thermodynamics & Green functions
   - Bethe-Salpeter & parquet equations
2. Thermodynamically consistent many-body parquet approach
   - Schwinger-Dyson equation and Ward identity generally
   - Perturbing the self-energy
   - 2P approach - Linearized Ward identity and Schwinger-Dyson equation
3. Thermodynamic consistency in disordered systems
   - 2P reducibility and parquet equations in disordered systems
   - Ward identity and the full 2P vertex
4. Conclusions

Equilibrium Hamiltonian & general perturbation

Equilibrium hamiltonian: Tight-binding description

$$\widehat{H} = \sum_{\mathbf{k}\sigma} \epsilon(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{i}\sigma} V_i \widehat{n}_{\mathbf{i}\sigma} + U \sum_{\mathbf{i}} \widehat{n}_{\mathbf{i}\uparrow} \widehat{n}_{\mathbf{i}\downarrow}$$

General perturbation: Normal & anomalous terms

$$\begin{aligned}\widehat{H}_{ext} = \int d1d2 \Bigg\{ & \sum_\sigma \eta^{||}_\sigma(1,2) c^\dagger_\sigma(1) c_\sigma(2) \quad \ldots \text{conserves charge \& spin} \\ & + \left[\eta^\perp(1,2) c^\dagger_\uparrow(1) c_\downarrow(2) + \bar{\eta}^\perp(1,2) c^\dagger_\downarrow(2) c_\uparrow(1)\right] \quad \ldots \text{conserves charge} \\ & + \sum_\sigma \left[\bar{\xi}^{||}_\sigma(1,2) c_\sigma(1) c_\sigma(2) + \xi^{||}_\sigma(1,2) c^\dagger_\sigma(1) c^\dagger_\sigma(2)\right] \quad \ldots \text{conserves spin} \\ & + \left[\bar{\xi}^\perp(1,2) c_\uparrow(1) c_\downarrow(2) + \xi^\perp(1,2) c^\dagger_\downarrow(2) c^\dagger_\uparrow(1)\right] \ldots \text{changes charge \& spin} \Bigg\}\end{aligned}$$

Thermodynamics & Green functions - unrenormalized

- Thermodynamic potential with external sources (weak non-equilibrium)

$$\Omega[G^{(0)-1}, H] = -\beta^{-1} \log \mathrm{Tr} \left[\exp \left\{ -\beta \left(\widehat{H} + \widehat{H}_{ext} - \mu \widehat{N} \right) \right\} \right]$$

unperturbed 1P Green function $G^{(0)-1} = [i\omega_n - \epsilon(\mathbf{k}) - \mu]$

- 1P Green function

$$G_{\sigma\sigma'}(\mathbf{k}, \mathbf{k}'; \tau, \tau') = -\frac{1}{\hbar} \mathrm{Tr} \left\{ \widehat{\rho}_H \, \mathcal{T} \left[c^{\dagger}_{\mathbf{k}\sigma}(\tau) c_{\mathbf{k}'\sigma'}(\tau') \right] \right\} = \frac{\delta \Omega[G^{(0)-1}, H]}{\delta G^{(0)-1}(\mathbf{k}, \tau; \mathbf{k}', \tau')}$$

$$\widehat{\rho}_H = \exp \left\{ -\beta \widehat{H} \right\} / \mathrm{Tr} \exp \left\{ -\beta \widehat{H} \right\}$$

- 2P Green function

$$G_2(12, 34) = -\frac{1}{\hbar^2} \mathrm{Tr} \left\{ \widehat{\rho}_H \, \mathcal{T} \left[\widehat{\psi}(1) \widehat{\psi}(3) \widehat{\psi}(4)^{\dagger} \widehat{\psi}(2)^{\dagger} \right] \right\}$$

$$1 = (\mathbf{R}_1, \tau_1) \ldots$$

Green functions from a renormalized functional (Baym & Kadanoff)

- Renormalized generating functional -- "Legendre transform" of the thermodynamic potential

$$\Phi[G,H] = \Omega[G^{(0)-1},H] - \int d\,\bar{1}\left(G^{(0)-1}(1,\bar{1}) - G^{-1}(1,\bar{1})\right) G(\bar{1},1')$$

- 1P Green function and self-energy (equilibrium)

$$G^{\alpha}(12) = \left.\frac{\delta\Phi[G,H]}{\delta H_{\bar{\alpha}}(2,1)}\right|_{H=0}, \qquad \Sigma^{\alpha}(12) = \frac{\delta\Phi[G,0]}{\delta G_{\bar{\alpha}}(2,1)}$$

- 2P Green and vertex functions (equilibrium)

$$G^{(2)\alpha}(13,24) = \left.\frac{\delta^2\Phi[G,H]}{\delta H_{\alpha}(4,3)\delta H_{\bar{\alpha}}(2,1)}\right|_{H=0}, \Lambda^{\alpha}(13,24) \quad = \frac{\delta^2\Phi[G,0]}{\delta G_{\alpha}(4,3)\delta G_{\bar{\alpha}}(2,1)}$$

Fundamental relations between 1P & 2P GF

- Dyson equation

$$G^{(0)-1}(1,2) - G^{-1}(1,2) = \Sigma^{\alpha}(12) = \left.\frac{\delta\Phi[G,H]}{\delta G_{\alpha}(2,1)}\right|_{H=0}$$

- Schwinger-Dyson equation -- projection of Schrödinger equation

$$\Sigma_{\sigma}(k) = \frac{U}{\beta N}\sum_{k'} G_{-\sigma}(k')\left[1 - \frac{1}{\beta N}\sum_{q}\Gamma_{\sigma-\sigma}(k,k';q)G_{\sigma}(k+q)G_{-\sigma}(k'+q)\right]$$

- Bethe-Salpeter equations

$$\Gamma(k,k';q) = \Lambda^{\alpha}(k,k';q) - \left[\Lambda^{\alpha}GG\odot\Gamma\right](k,k';q)$$

- Generalized Ward identity (thermodynamic consistency)

$$\Lambda^{\alpha}(13,24) = \frac{\delta\Sigma^{\alpha}(1,2)}{\delta G^{\alpha}(4,3)}$$

- SD & WI hold simultaneously in full exact but none approximate (even asymptotically exact) theory

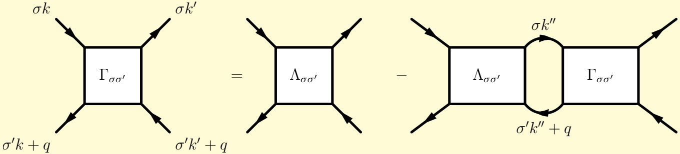
Necessity to control directly 2P functions

- Critical behavior and phase transitions due to singularities in Bethe-Salpeter equations
- Two-particle self-consistency needed to eliminate unphysical (non-integrable) singularities
- Stable equilibrium state in the critical region -- full control of 2P functions necessary
- BK approach does not work -- 2P vertex not explicitly known for the given self-energy (2P singularities not controlled)
- Inverse procedure to BK:
  - Direct (diagrammatic) approximation for 2P vertices
  - Introduce a two-particle self-consistency (when needed)
  - Construct a self-energy to the given 2P vertex
  - Ward identity generally lost

Thermodynamic consistency -- Ward identity to be reintroduced (macroscale)

Bethe- Salpeter equation - electron-hole channel

- Multiple simultaneous scatterings -- electron-hole ladder



- Conserving (bosonic) transfer (four)momentum: $k - k'$

$$\Gamma_{\sigma\sigma'}(k, k'; q) = \Lambda^{eh}_{\sigma\sigma'}(k, k'; q) - \frac{1}{\beta\mathcal{N}} \sum_{q''} \Lambda^{eh}_{\sigma\sigma'}(k, k'; q'')$$

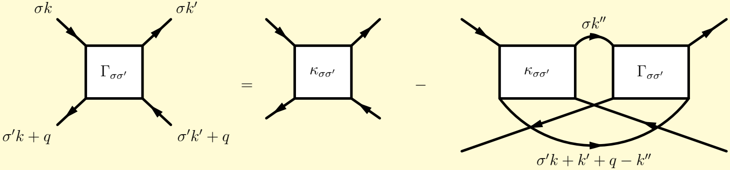
$$\times\, G_{\sigma}(k + q'') G_{\sigma'}(k' + q'') \Gamma_{\sigma\sigma'}(k + q'', k' + q''; q - q'')$$

- Decomposition of the full vertex: All = irreducible $\cup$ reducible (diagrams)

$$\Gamma_{\sigma\sigma'} = \Lambda^{eh}_{\sigma\sigma'} + \mathcal{K}^{eh}_{\sigma\sigma'}$$

Bethe- Salpeter equation - electron-electron channel

- Multiple simultaneous scatterings -- electron-electron ladder



- Conserving (bosonic) transfer (four)momentum: $k + k' + q$

$$\Gamma_{\sigma\sigma'}(k, k'; q) = \Lambda^{ee}_{\sigma\sigma'}(k, k'; q) - \frac{1}{\beta\mathcal{N}} \sum_{q''} \Lambda^{ee}_{\sigma\sigma'}(k, k' + q''; q - q'')$$

$$\times\ G_\sigma(k + q - q'') G_{\sigma'}(k' + q'') \Gamma_{\sigma\sigma'}(k + q - q'', k'; q'')$$

- Decomposition of the full vertex: All $=$ irreducible $\cup$ reducible (diagrams)

$$\Gamma_{\sigma\sigma'} = \Lambda^{ee}_{\sigma\sigma'} + \mathcal{K}^{ee}_{\sigma\sigma'}$$

Vertex functions -- Parquet approach (two channels)

- Basic concept: 2P reducibility -- not uniquely defined
- Channel-dependent decompositions of the full vertex:
  $\Gamma_{\sigma\sigma'} = \Lambda^{ee}_{\sigma\sigma'} + \mathcal{K}^{ee}_{\sigma\sigma'} = \Lambda^{eh}_{\sigma\sigma'} + \mathcal{K}^{eh}_{\sigma\sigma'}$
- Fully irreducible vertex (diagrammatically): $\mathcal{I} = \Lambda^{eh} \cap \Lambda^{ee}$
- Existence (applicability) of the parquet decomposition:

$$\mathcal{K}^{ee} \cap \mathcal{K}^{eh} = \emptyset$$

- Fundamental parquet decomposition:

$$\Gamma = \mathcal{I} \cup \mathcal{K}^{ee} \cup \mathcal{K}^{eh} = \Lambda^{eh} \cup \Lambda^{ee} \setminus \mathcal{I} = \mathcal{I} + \mathcal{K}^{eh} + \mathcal{K}^{ee} = \Lambda^{ee} + \Lambda^{eh} - \mathcal{I}$$

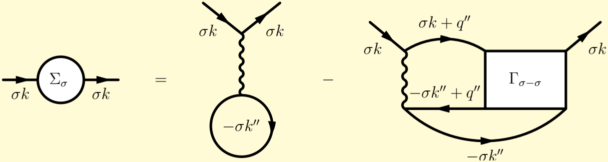
- Parquet equations: Bethe-Salpeter equations with Γ replaced by the fundamental parquet decomposition
- Input to parquet equations: fully irreducible vertex $\mathcal{I}$, G_σ
- Output: 2PIR vertices Λ^{eh} and Λ^{ee}
- No prescribed connection between Σ and Λ^{α} or Γ

Parquet equations for interacting systems

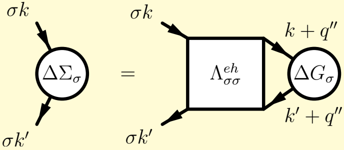
- Non-locality in space & time -- parquet equations also locally
- 1P propagators in the parquet equations:
  Self-energy from Schwinger-Dyson equation
  - Thermodynamic consistency not guaranteed: Singularity in the BSE does not break the symmetry of the self-energy
  - Parquet equations (2P approach) break down beyond the critical point (singularity in BSE)!
- Thermodynamic consistency only via Ward identity
- Full WI cannot be resolved
- WI does not determine the self-energy (energy not conserved) (unlike disordered systems)

1P propagators & 2P vertices

- **Schwinger-Dyson equation**: Self-energy and the full 2P vertex



- **Ward identity**: Self-energy and 2P irreducible vertex



$$\Delta\Sigma_\sigma(k,k') = \Sigma_\sigma(k) - \Sigma_\sigma(k'),\ \Delta G_\sigma(k,k') = G_\sigma(k) - G_\sigma(k')$$

One can never fulfill both identities!

Two-particle criticality

- Transition to an ordered state via a critical point -- divergence in a Bethe-Salpeter equation
- Magnetic order -- electron hole bubbles (local irreducible vertex)

$$\Gamma_{\uparrow\downarrow}(k,k';q) = \frac{\Lambda^U_{\uparrow\downarrow}}{1 - \Lambda^U_{\uparrow\downarrow}\chi_{\downarrow\downarrow}(k-k')\Lambda^U_{\downarrow\uparrow}\chi_{\uparrow\uparrow}(k-k')} \qquad (1)$$

with $\chi_{\sigma\sigma'}(q) = (\beta N)^{-1}\sum_k G_\sigma(k)G_{\sigma'}(k+q)$

- Critical point: $\Lambda^U_{\uparrow\downarrow}\chi_{\downarrow\downarrow}(0) = -1$

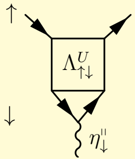
How to treat the theory beyond the critical point?

- Emergence of magnetic order - spin-polarized self-energy
- Only in thermodynamically consistent theories

Perturbed self-energy - magnetic ordering

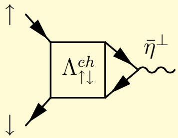
- Repulsive particle interaction -- electron-hole scattering dominant
- Linear-response theory -- unperturbed Λ^{eh} (Λ^{U}) determines the self-energy
- Longitudinal magnetic order (*eh* bubbles): normal self-energy

$\Sigma_{\uparrow}(\eta^{\parallel}_{\downarrow}) \doteq$



- Transversal (spin flip) magnetic order (*eh* ladders): self-energy anomalous only in the spin-polarized state

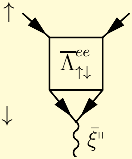
$\Sigma_{\uparrow\downarrow}(\bar{\eta}^{\perp}) \doteq$



Perturbed self-energy - superconducting ordering

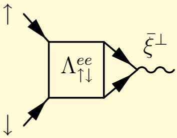
- Attractive particle interaction -- electron-electron scattering dominant
- Linear-response theory -- unperturbed Λ^{ee} determines the self-energy
- Triplet superconducting order (ee bubbles): anomalous self-energy & anomalous vertex (does not conserve spin)

$$\Sigma_{\uparrow}(\bar{\xi}^{\shortparallel}_{\downarrow}) \doteq$$



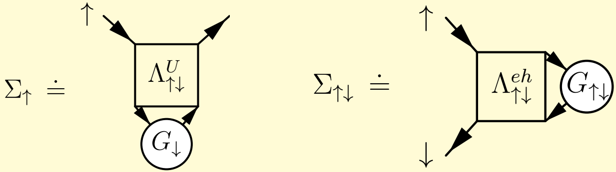
- Singlet superconducting order (ee ladders): anomalous self-energy

$$\Sigma_{\uparrow\downarrow}(\bar{\xi}^{\perp}) \doteq$$



Linearized Ward identity

- WI resolved w.r.t. symmetry-breaking field - only normal component
- Linearized WI in the external magnetic field (longitudinal & transversal)



- Mathematical expressions

$$\Sigma_\uparrow(k) = \frac{1}{\beta N}\sum_q \Lambda^U_{\uparrow\downarrow}(k,k;q)G_\downarrow(k+q)$$

$$\Sigma_{\uparrow\downarrow}(k) = \frac{1}{\beta N}\sum_{k'} \Lambda^{eh}_{\uparrow\downarrow}(k,k';0)G_{\uparrow\downarrow}(k')$$

- Vertex depends quadratically on the perturbing (magnetic) field

SDE in the thermodynamically consistent approach

- Linearized WI: symmetry of the self-energy gets broken at the divergence in the BSE for a zero eigenvalue of

$$M_{k,k'} = \delta_{k,k'} + \Lambda_{\uparrow\downarrow}(k,k';0)G_{\uparrow}(k')G_{\downarrow}(k')$$

- 1P propagators should use Σ from LWI in all equations with 2P functions: BSE, SDE

Schwinger-Dyson equation with $\Gamma_{\sigma\sigma'}$ and G_σ from the parquet equations determines the physical (thermodynamic) self-energy

$$\Sigma_{\uparrow}(k) = \frac{U}{\beta N}\sum_{k'} G_{\downarrow}(k')\left[1 - \frac{1}{\beta N}\sum_{k'} G_{\downarrow}(k'')G_{\uparrow}(k+k'-k'') \times \Gamma_{\uparrow\downarrow}(k',k'';k-k'')\right]$$

SDE in the thermodynamically consistent approach

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$$\Sigma_{\uparrow}(k) = \frac{U}{\beta N}\sum_{k'} G_{\downarrow}(k')\left[1 - \frac{1}{\beta N}\sum_{k'} G_{\downarrow}(k'')G_{\uparrow}(k+k'-k'') \right.$$
$$\left. \times \Gamma_{\uparrow\downarrow}(k',k'';k-k'')\right]$$

Parquet equations for disordered electrons

- Quenched disorder & no ee interaction
  -- no dynamics, energy conserved
- Equilibrium thermodynamics
  -- renormalization of the dispersion relation
- Non-equilibrium -- Linear Response Theory (Kubo formalism)
  - Equilibrium two-particle Green function needed
  - Multiple scattering on impurities
    - Local scatterings -- mean field (CPA)
    - Nonlocal scatterings -- electron-electron and electron-hole simultaneous propagation distinguishable
  - Two-particle self-consistency -- via parquet equations (beyond CPA)
  - Vertex function not fully compatible with the Ward identity
- Thermodynamic consistency -- corrections to the vertex from the parquet equations so that Ward identity is obeyed

Model description of scatterings on impurities

Noninteracting lattice electrons in a random lattice (impurities) in tight-binding representation:

$$\widehat{H}_{AD} = \sum_{<ij>} t_{ij} c_i^\dagger c_j + \sum_i V_i c_i^\dagger c_i$$

Disorder distribution (site independent):

$$\langle X(V_i)\rangle_{av} = \int_{-\infty}^{\infty} dV \rho(V) X(V)$$

binary alloy: $\rho(V) = c\delta(V - \Delta) + (1 - c)\delta(V + \Delta)$

Quenched disorder: Averaged free energy (thermodynamics)

$$F_{av} = -k_B T \left\langle \ln \mathrm{Tr} \exp\left\{-\beta \widehat{H}_{AD}(t_{ij}, V_i)\right\}\right\rangle_{av}$$

Good for thermodynamics and averaged one-electron functions, no information on transport and dynamical quantities

Two-particle reducibility and parquet decomposition

- Distinction between electron and holes necessary for parquet equations -- non locality in time or space
- Static theory -- non-local scatterings (beyond local mean-field)
- Expansion beyond MFT: non-local (off-diagonal) 1PGF

$$\overline{G}(E_{\pm}, \mathbf{k}) = \frac{1}{N}\sum_{\mathbf{k}'} \frac{N\delta_{\mathbf{k},\mathbf{k}'} - 1}{E \pm i0^{+} - \epsilon(\mathbf{k}') - \Sigma(E \pm i0^{+}, \mathbf{k})}$$

- Distinguishable two-particle reducibility
- Local diagrams -- fully irreducible
- Electron-hole simultaneous propagation: $\overline{G}(E_{+}, \mathbf{k})\overline{G}(E_{-}, \mathbf{q} + \mathbf{k})$
- Electron-electron simultaneous propagation: $\overline{G}(E_{+}, \mathbf{k})\overline{G}(E_{-}, \mathbf{q} - \mathbf{k})$

Bethe-Salpeter and the parquet equations (no WI)

- BS equation with multiple nonlocal *eh* scatterings

$$\Gamma_{\mathbf{k}\mathbf{k}'}(E_+,E_-;\mathbf{q}) = \overline{\Lambda}^{eh}_{\mathbf{k}\mathbf{k}'}(E_+,E_-;\mathbf{q}) + \frac{1}{N}\sum_{\mathbf{k}''}\overline{\Lambda}^{eh}_{\mathbf{k}\mathbf{k}''}(E_+,E_-;\mathbf{q}) \times \overline{G}(E_+,\mathbf{k}'')\overline{G}(E_-,\mathbf{k}''+\mathbf{q})\Gamma_{\mathbf{k}''\mathbf{k}'}(E_+,E_-;\mathbf{q})$$

- BS equation with multiple nonlocal *ee* scatterings

$$\Gamma_{\mathbf{k}\mathbf{k}'}(E_+,E_-;\mathbf{q}) = \overline{\Lambda}^{ee}_{\mathbf{k}\mathbf{k}'}(E_+,E_-;\mathbf{q}) + \frac{1}{N}\sum_{\mathbf{k}''}\overline{\Lambda}^{ee}_{\mathbf{k}\mathbf{k}''}(E_+,E_-;\mathbf{q}+\mathbf{k}'-\mathbf{k}'') \times \overline{G}(E_+,\mathbf{k}'')\overline{G}(E_-,\mathbf{Q}-\mathbf{k}'')\Gamma_{\mathbf{k}''\mathbf{k}'}(E_+,E_-;\mathbf{q}+\mathbf{k}-\mathbf{k}'')$$

$$\mathbf{Q} = \mathbf{q}+\mathbf{k}+\mathbf{k}'$$

- Parquet equation

$$\Gamma_{\mathbf{k}\mathbf{k}'}(E_+,E_-;\mathbf{q}) = \overline{\Lambda}^{eh}_{\mathbf{k}\mathbf{k}'}(E_+,E_-;\mathbf{q}) + \overline{\Lambda}^{ee}_{\mathbf{k}\mathbf{k}'}(E_+,E_-;\mathbf{q}) - \mathcal{I}_{\mathbf{k}\mathbf{k}'}(E_+,E_-;\mathbf{q})$$

Fully irreducible vertex $\mathcal{I}$ - contains all local and multiply crossed diagrams

2P electron-hole symmetry - missing in CPA

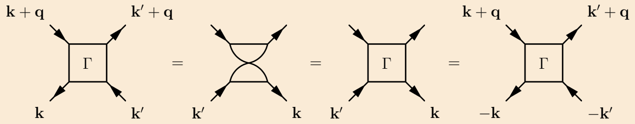
- Charge & time reflection (bipartite lattice)

$$G(\mathbf{k}, z) = G(-\mathbf{k}, z)$$

- Two-particle symmetry: Full vertex

$$\Gamma_{\mathbf{k}\mathbf{k}'}(z_+, z_-; \mathbf{q}) = \Gamma_{\mathbf{k}\mathbf{k}'}(z_+, z_-; -\mathbf{Q}) = \Gamma_{-\mathbf{k}'-\mathbf{k}}(z_+, z_-; \mathbf{Q})$$

$(\mathbf{Q} = \mathbf{q} + \mathbf{k} + \mathbf{k}')$



- Irreducible vertices: Symmetry transformation

$$\bar{\Lambda}^{ee}_{\mathbf{k}\mathbf{k}'}(z_+, z_-; \mathbf{q}) = \bar{\Lambda}^{eh}_{\mathbf{k}\mathbf{k}'}(z_+, z_-; -\mathbf{Q}) = \bar{\Lambda}^{eh}_{-\mathbf{k}'-\mathbf{k}}(z_+, z_-; \mathbf{Q})$$

Ward identity

- 2P vertex from parquet equations (or other 2P constructions) does not fully obey Ward identity
- 1P propagators in 2P approaches are input
- Vollhardt-Wölfle identity (continuity equation) ($\mathbf{k}_\pm = \mathbf{k} \pm \mathbf{q}/2$)

$$\Sigma(z_+, \mathbf{k}_+) - \Sigma(z_-, \mathbf{k}_-) = \frac{1}{N} \sum_{\mathbf{k}'} \Lambda_{\mathbf{k}\mathbf{k}'}(z_+, z_-; \mathbf{q}) \left[G(z_+, \mathbf{k}'_+) - G(z_-, \mathbf{k}'_-) \right]$$

$G^{(2)} = GG + \Lambda GG \star G^{(2)}$ -- Bethe-Salpeter equation

- WI guaranteed in 2P approaches (parquets) only for $\omega = 0$ and $q = 0$

$$\boxed{\Lambda_{\mathbf{k}\mathbf{k}'}(z_+, z_-; \mathbf{q}) \neq \overline{\Lambda}_{\mathbf{k}\mathbf{k}'}(z_+, z_-; \mathbf{q})}$$

Vertex for the Ward identity & self-energy

- Self-energy in the 1P propagator of the parquet equations ($E_\pm = E \pm i0^+$)

$$\Im\Sigma(E_+, \mathbf{k}) = \frac{1}{N} \sum_{\mathbf{k}'} \Lambda_{\mathbf{k}\mathbf{k}'}(E_+, 0) \Im G(E_+, \mathbf{k}')$$

$$\Re\Sigma(E_+, \mathbf{k}) = \Sigma_\infty + P \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\Im\Sigma(\omega_+, \mathbf{k})}{\omega - E}$$

- Vertex Λ -- irreducible also locally (*eh* and *ee* processes indistinguishable)
- Irreducible vertex for the Ward identity from the vertex of the parquet equations $\overline{\Lambda}$

$$\Lambda_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = \bar{\Lambda}_{\mathbf{k}\mathbf{k}''}(\mathbf{q}) + \frac{1}{N} \sum_{\mathbf{k}''} \bar{\Lambda}_{\mathbf{k}\mathbf{k}''}(\mathbf{q}) \left[G_+(\mathbf{k}'') \langle G_- \rangle + \langle G_+ \rangle G_-(\mathbf{k}_-) - \langle G_+ \rangle \langle G_- \rangle \right] \Lambda_{\mathbf{k}''\mathbf{k}'}(\mathbf{q})$$

$$\langle G_\pm \rangle = N^{-1} \sum_{\mathbf{k}} G(E_\pm, k)$$

- New quantities to define a vertex compatible with WI

$$\Delta G_{\mathbf{k}}(\omega,\mathbf{q}) = G(E_+,\mathbf{k}_+) - G(E_-,\mathbf{k}_-)$$
$$\Delta\Sigma_{\mathbf{k}}(\omega,\mathbf{q}) = \Sigma_{\mathbf{k}_+}(E_+,\mathbf{k}_+) - \Sigma(E_-,\mathbf{k}_-)$$
$$R_{\mathbf{k}}(\omega,\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{k}'}\Lambda_{\mathbf{k}\mathbf{k}'}(\omega,\mathbf{q})\Delta G_{\mathbf{k}'}(\omega,\mathbf{q}) - \Delta\Sigma_{\mathbf{k}}(\omega,\mathbf{q})$$
$$\left\langle \Delta G(\omega,\mathbf{q})^2\right\rangle = \frac{1}{N}\sum_{\mathbf{k}}\Delta G_{\mathbf{k}}(\omega,\mathbf{q})^2$$

$E_\pm = E \pm \omega/2 \pm i0^+$, $\mathbf{k}_\pm = \mathbf{k} \pm \mathbf{q}/2$

- BS equation for a thermodynamically consistent (physical) 2P vertex Γ

$$\frac{1}{N}\sum_{\mathbf{k}''}\left\{\delta_{\mathbf{k},\mathbf{k}''} - \left[\Lambda_{\mathbf{k}\mathbf{k}''} - \frac{\Delta G_{\mathbf{k}} R_{\mathbf{k}''}}{\langle\Delta G^2\rangle} - \frac{R_{\mathbf{k}}\Delta G_{\mathbf{k}''}}{\langle\Delta G^2\rangle} + \langle R\Delta G\rangle\frac{\Delta G_{\mathbf{k}}\Delta G_{\mathbf{k}''}}{\langle\Delta G^2\rangle^2}\right]\right.$$
$$\left.\times G_{\mathbf{k}'_+}G_{\mathbf{k}'_-}\right\}\Gamma_{\mathbf{k}''\mathbf{k}'} = \Lambda_{\mathbf{k}\mathbf{k}'} - \frac{\Delta G_{\mathbf{k}} R_{\mathbf{k}'}}{\langle\Delta G^2\rangle} - \frac{R_{\mathbf{k}}\Delta G_{\mathbf{k}'}}{\langle\Delta G^2\rangle} + \langle R\Delta G\rangle\frac{\Delta G_{\mathbf{k}}\Delta G_{\mathbf{k}'}}{\langle\Delta G^2\rangle^2}$$

Conclusions I

Parquet approach -- many-body & general

- Applicability of parquet approach
  -- distinguishability of electrons and holes
- Dynamical or nonlocal scatterings
- Intermediate coupling -- a divergence in a BS equation (RPA pole)
- To go beyond the pole -- 1P order parameter is to be introduced
- Thermodynamic consistency between 1P propagators & 2PI vertex in parquet equations
- Linearized Ward identity -- self-energy in 1P propagators in parquet equations
- Schwinger-Dyson equation -- determines the physical self-energy (not self-consistent in parquet equations)

Conclusions II

Parquet approach -- disordered systems

- Parquet approach only to nonlocal vertices
  -- beyond mean field (CPA)
- Electron-hole symmetry on 2P level leads to
  a single nonlinear integral equation
- 2P vertex does not obey Ward identity
- Ward identity induces restriction of the irreducible vertex in 2P space
- Corrections to 2PIR vertex to restore WI
- Full 2P vertex from a Bethe-Salpeter equation with a restricted irreducible vertex and WI corrections
- Physical quantities from the full vertex obeying WI